

EXTRA PROBLEMS #5

SOLUTIONS

In this assignment, we consider again rational and irrational numbers.

We begin with a warm-up exercise (that will also be useful).

Exercise 0.1. Suppose $f : A \rightarrow B$ is onto (here the domain of f is A and the codomain is B). Also suppose that $g : B \rightarrow C$ is onto. Recall that $g \circ f$ (the composition of the functions g and f) is the function associated to the rule $(g \circ f)(x) = g(f(x))$. Prove that $g \circ f : A \rightarrow C$ is also onto.

Proof. Choose $c \in C$ arbitrary. We need to show there is an $a \in A$ such that $g(f(a)) = c$. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. Then

$$g(f(a)) = g(b) = c$$

as desired. □

Exercise 0.2. First show that there exists a function $f : \mathbb{N} \rightarrow \mathbb{Q}_{>0}$ that is onto. Here $\mathbb{Q}_{>0}$ is used to denote the set of positive rational numbers.

Hint: We are trying to organize the rational numbers into an infinite list. Consider arranging the positive rational numbers like the diagram below.

$$\begin{array}{ccccccc} 1/1 & 1/2 & 1/3 & 1/4 & \dots & & \\ 2/1 & 2/2 & 2/3 & 2/4 & \dots & & \\ 3/1 & 3/2 & 3/3 & 3/4 & \dots & & \\ 4/1 & 4/2 & 4/3 & 4/4 & \dots & & \\ \dots & \dots & \dots & \dots & \dots & & \end{array}$$

Proof. We simply need to arrange all the positive rational numbers into a list (where we allow repeats). Once we have done this, we define $f(n)$ to be the n th entry in the list. We start with $(1/1), (1/2), (2/1), (1/3), (2/2), (3/1), (1/4), \dots$. We repeat this pattern by going down the diagonal to the left, until we run out of numbers and then starting on the next diagonal. Every positive rational number eventually appears in this list, so we are done. □

Exercise 0.3. Show that there is an onto function $g : \mathbb{N} \rightarrow \mathbb{Q}$.

Hint: First construct an onto function from \mathbb{Z} to \mathbb{Q} , then use a previous extra math assignment to find an onto function from \mathbb{N} to \mathbb{Z} . Conclude by applying the first exercise.

Proof. We first define an onto function $j : \mathbb{Z} \rightarrow \mathbb{Q}$ by

$$j(n) = \begin{cases} f(n) & n > 0 \\ 0 & n = 0 \\ -f(|n|) & n < 0 \end{cases}$$

By extra math assignment #3, there exists an onto function $k : \mathbb{N} \rightarrow \mathbb{Z}$. Thus the composition $j \circ k : \mathbb{N} \rightarrow \mathbb{Q}$ is surjective by the first exercise. □

In Set #3 we showed that every interval contains infinitely many rational numbers. Use the same kind of ideas to

Exercise 0.4. Show that

- Between two rational numbers there is an irrational number.
- Between two irrational numbers there is a rational number.

Proof. The proof of the second part was already done in Extra Problems #3, Exercise 0.4 (in fact, we showed there were *infinitely* many rational numbers between *any* two numbers). We will just do the first part.

Suppose p and q are rational numbers with $p < q$. We want to find an irrational number between them. Let n be an integer large enough so that

$$n(q - p) > 2.$$

We can appeal to the decimal expansion of $q - p$ to prove the existence of such an n . Simply choose $n = 10^{k+1}$ where $k = 1$ if $q - p \geq 1$ and otherwise choose k to be larger than the number of zeros after the decimal point (before the first non-zero number).

Then we note between nq and np there must be at least two integers. Call them a and b with $a < b$. Consider the number $z = \sqrt{3} - 1 + a$. This number is certainly between a and b , and thus also between nq and np . We wish to show that z/n is irrational.

Suppose it was rational, then $z/n = (\sqrt{3} - 1 + a)/n = l/k$ for integers l, k . Then $\sqrt{3} = nl/k + 1 - a = \frac{nl+k-ka}{k}$ so that $\sqrt{3}$ is a rational number. This is impossible since we know $\sqrt{3}$ is irrational by Extra Problems #3. Thus we know that z is irrational. But then since $np < z < nq$, we have $p < z/n < q$, and we have found our irrational number, z/n . \square

This exercise can give a feeling that there are as many irrationals than rationals. Since both sets have infinitely many numbers, this is true in some sense. However, sets with infinitely many numbers can be quite different and we will argue in what follows that in some sense there are more irrationals than rationals.

Theorem 0.5 (Cantor). *Show that there is no function $f: \mathbb{N} \rightarrow [0, 1)$ that is onto.*

Our next goal is to prove this theorem.

We will prove the theorem by contradiction. Suppose we have such a function f . Write each $f(n)$ as a decimal expansion and arrange them in a list like so,

$$\begin{array}{rcll} f(1) = & 0 & . & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & \dots \\ f(2) = & 0 & . & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \dots \\ f(3) = & 0 & . & d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & \dots \\ f(4) = & 0 & . & d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & \dots \\ & & & \dots & & & & & \end{array}$$

We also assume that each decimal expansion does *NOT* have an infinite trail of 9s at the end. (Remember $0.6499999999\dots = 0.65$). Recall that once we make this assumption, the decimal expansion of a number is unique (that is, there is only one way to do it).

Exercise 0.6. Prove Theorem 0.5.

Hint: Draw a diagonal line down and to the right, starting at the digit d_{11} . By considering the digits on that diagonal, explain why you can construct a new decimal number that is

- Not on the list.
- Inside the interval $(0, 1)$.
- Does not have any 9s at all in it (let alone an infinite trail of 9s).

Proof. Suppose that $f: \mathbb{N} \rightarrow [0, 1)$ was onto. We define a new number e via decimal expansion $e = 0.e_1e_2e_3e_4e_5\dots$ where each e_n is chosen so that $e_n \neq d_{nn}$ and $e_n \neq 9$. Then note that e cannot be on the list, because if it was, say $e = f(m)$ for some m , we know

that e and $f(m)$ have the same decimal expansion. But they don't, since $e_m \neq f_{mm}$. Thus we conclude that the function f cannot be onto because we found a number $e \in [0, 1)$ such which f never "hits". \square

Exercise 0.7. Give an example of an onto function $h : \mathbb{R} \rightarrow [0, 1)$.

Proof. Define h to be the function which takes a number, views it as
 (integer).(decimal expansion)

and outputs

$$0.(\text{decimal expansion})$$

The function h is clearly onto $[0, 1)$ since it is the identity function on $[0, 1)$. \square

Exercise 0.8. Prove that there is no onto function $g : \mathbb{N} \rightarrow \mathbb{R}$.

Hint: Suppose there was, then use the previous two exercises and exercise 0.1 to prove this is impossible.

Proof. Suppose that g was onto, we will aim for a contradiction. Compose with h to get a function $h \circ g : \mathbb{N} \rightarrow [0, 1)$. Note that this function is onto by the first exercise of this set. But this is impossible by Theorem 0.5. \square

We are now ready to show that there are "more" irrationals than rationals. But first a bit of notation, $\mathbb{R} \setminus \mathbb{Q}$ is just an expression that means *all the real numbers that are not rational*. That is $\mathbb{R} \setminus \mathbb{Q}$ is equal to the set of irrational numbers.

Exercise 0.9. Show that there is no function $f : \mathbb{Z} \rightarrow \mathbb{R} \setminus \mathbb{Q}$ that is onto.

Hint: Use Exercise 0.5 together with Exercise 0.3 and also the last problem of Set 3.

Proof. Again, suppose there was such a function, we will aim for a contradiction. Let $l : \mathbb{N} \rightarrow \mathbb{Z}$ be an onto function which makes $f \circ l : \mathbb{N} \rightarrow \mathbb{R} \setminus \mathbb{Q}$ an onto function. Construct a new function $k : \mathbb{Z} \rightarrow \mathbb{R}$.

$$k(n) = \begin{cases} f(l(n)) & n > 0 \\ 65.12 & n = 0 \\ g(|n|) & n < 0 \end{cases}$$

where g is the function from Exercise 0.3. Since $f \circ l : \mathbb{N} \rightarrow \mathbb{R} \setminus \mathbb{Q}$ hits all the irrational numbers, and $g : \mathbb{N} \rightarrow \mathbb{Q}$ hits all the rational numbers, our new function $k : \mathbb{Z} \rightarrow \mathbb{R}$ is also onto. But then composing with l yet again, we get an onto function $k \circ l : \mathbb{N} \rightarrow \mathbb{R}$, which is impossible by the previous exercise. \square