EXTRA PROBLEMS #9

DUE: TUESDAY DECEMBER 11TH

A function need not be continuous to have a definite integral. The concept of integral can be developed when the functions under consideration are only bounded. Then, however, function need not to always have a definite integral. In this set we develop this theory.

Definition 0.1 (Upper and lower sums)**.** Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and $a = x_0 < x_1 < \cdots < x_n = b$ a subdivision of $[a,b]$. We set the *upper sum of f with respect to $x_0 < \cdots < x_n$* to be

$$S(f : x_0, x_1, \ldots, x_n) = \sum_{k=1}^{n} \left(\sup_{[x_{k-1},x_k]} f \right) (x_k - x_{k-1})$$

and *the lower sum* to be

$$s(f : x_0, x_1, \ldots, x_n) = \sum_{k=1}^{n} \left(\inf_{[x_{k-1},x_k]} f \right) (x_k - x_{k-1}).$$

Definition 0.2. A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if for every $\varepsilon > 0$ there exists a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ so that

$$|S(f; x_0, \ldots, x_n) - s(f; x_0, \ldots, x_n)| < \varepsilon.$$

The Riemann integral of f is set to be the unique value satisfying

$$s(f; x_0, \ldots, x_n) \le \int_a^b f \le S(f; x_0, \ldots, x_n).$$

for all subdivisions $a = x_0 < x_1 < \cdots < x_n = b$. (The existence of such number is proved in the last exercise.)

Exercise 0.3. Let $f\colon [-1,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}.$$

Show that f is Riemann integrable. What is the value of $\int_{-1}^{1} f(x)\,\mathrm{d}x$?

Exercise 0.4. Let $f\colon [-1,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1, & |x| = 1/n \text{ for} n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

Show that f is Riemann integrable. What is the value of $\int_{-1}^{1} f(x)\,\mathrm{d}x$?

Hint to first two problems: Find good subdivisions.

Exercise 0.5. Let $f\colon [-1,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin Q \end{cases}.$$

Show that f is NOT Riemann integrable.

Hint: You need to consider all subdivisions. Look max's and min's.

In the following problems, we show that for continuous we recover the (usual) theory of definite integration.

Exercise 0.6. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Show that if $a < x_1 < b$ then $S(f; a, x_1, b) \leq S(f; a, b)$ and $s(f; a, x_1, b) \geq s(f; a, b)$.

Exercise 0.7. Use a problem from the previous exercise set to show that if subdivision $a = y_0 < y_1 < \cdots < y_m = b$ contains all the points of the subdivision $a = x_0 < x_1 < \cdots x_n = b$ then $S(f; y_0, \ldots, y_m) \leq S(f; x_0, \ldots, x_n)$ and $s(f; y_0, \ldots, y_m) \geq s(f; x_0, \ldots, x_n)$.

Exercise 0.8. Show that if $a = x_0 < \cdots < x_n = b$ and $a = y_0 < \cdots < y_m = b$ are subdivisions of $[a,b]$, then there exists a subdivision $a = z_0 < \cdots < z_k = b$ containing all the points of the other subdivisions.

Exercise 0.9. Show, using Problems 0.7 and 0.8, that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable then there exists a unique number I so that

$$s(f; x_0, \ldots, x_n) \leq I \leq S(f; x_0, \ldots, x_n)$$

for all subdivisions $a = x_0 < x_1 < \cdots x_n = b$.