EXTRA PROBLEMS #8

DUE: TUE. DEC 4TH

A function need not be continuous to have a definite integral. The concept of integral can be developed when the functions under consideration are only bounded. Then, however, function need not to always have a definite integral. In this set we develop this theory. See section 2-13 in the text for additional information.

Definition 0.1. Given a set of numbers E, we say that E *is bounded below* if there exists an $L \in \mathbb{R}$ such that $x \geq L$ for all $x \in E$, in this case we say that L *is a lower bound for* E. Similarly we say that E *is bounded above* if there exists a K such that $x \leq K$ for all $x \in E$, in this case we say that L *is an upper bound for* E. A set E is called *bounded* if it is both bounded above and bounded below.

Definition 0.2. Again, given a bounded below set of numbers E, we define the *greatest lower bound* of E to be the largest number L that is a lower bound for E. We denote this greatest lower bound by $\mathrm{glb}E$. Likewise, we define the *least upper bound* of E to be the smallest number K that is an upper bound for E. We denote this least upper bound by $\mathrm{lub}E$.

Implicit in the previous definition is the following fact about the real numbers (called the least upper bound axiom).

AXIOM: Every bounded below subset $E \subseteq \mathbb{R}$ has a greatest lower bound. Every bounded above subset $F \subseteq \mathbb{R}$ has a least upper bound.

For starters, consider the following least upper bound and greatest lower bound problems.

Exercise 0.3. (a) Let E be the set $\{1/2, 1/3, 1/4, 1/5, \ldots\}$. Find $\mathrm{glb}E$ and $\mathrm{lub}E$.
(b) Let E be the set $(-2, 0) \cup (0, 2)$. Find $\mathrm{glb}E$ and $\mathrm{lub}E$.
(c) Let E be the set $[-\pi, \pi] \cap \mathbb{Q}$. Find $\mathrm{glb}E$ and $\mathrm{lub}E$.

Exercise 0.4. Let E be a bounded non-empty subset of $\mathbb{R}$. Suppose that $F \subseteq E$ is also non-empty. Prove that $\mathrm{glb}F \geq \mathrm{glb}E$ and that $\mathrm{lub}F \leq \mathrm{lub}E$.

Exercise 0.5. Let E and F be bounded non-empty sets of $\mathbb{R}$. Find $\mathrm{glb}(E \cup F)$ and $\mathrm{lub}(E \cup F)$.

Exercise 0.6. Prove that $0 = \mathrm{glb}([0, 1])$ and that $1 = \mathrm{lub}([0, 1])$. (This is not particularly hard)

Recall that in extra problems #7, we defined what it meant for a subset of $\mathbb{R}$ to be connected (or disconnected).

Exercise 0.7. Prove that $[0, 1]$ is a connected subset of $\mathbb{R}$. (Note, this problem might be harder than some of the others, so it is ok to skip it if you don't see it right away).

Hint: Suppose that $[0, 1]$ was disconnected. We aim for a contradiction Thus $[0, 1] \subseteq U \cup V$ where $U \cap V = \emptyset$, $[0, 1]$ and U have points in common, $[0, 1]$ and V have points in common, and U and V are open subsets of $\mathbb{R}$.

Define a set E to be the set of points of $[0,1]$ that are not in U (in other words, $E = [0,1]\backslash([0,1] \cap U) = [0,1] \cap V$). Show that E is bounded and not empty, and let $L = \text{glb}E$. Prove that L is contained in $[0,1]$ (use a previous exercise). It will be helpful later if you can show that if $x \in [0,1]$ and $x < L$, then $x \in U$.

Now, there are two cases, either $L \in U$ or $L \in V$. Case 1: if L is in U, first show that $L \neq 1$. Now prove that L cannot be a least upper bound of E to get a contradiction (use the fact that U is open, so there is a little interval around L in U. Case 2: if L is in V and $L \neq 0$, use the fact that V is open and that $U \cap V = \emptyset$ to prove that L cannot be a least upper bound of E. Therefore the only possible case is that $L = 0$ and $L \in V$. Finally, define $F = \text{glb}[0,1] \cap U$ and let $M = \text{glb}F$. Argue by symmetry that $M = 0$ and $M \in U$. But then $0 = M = L \in U \cap V$.

Definition 0.8. A function $f\colon [a,b] \to \mathbb{R}$ is *bounded* if there exists $M > 0$ so that $|f(x)| \leq M$ for all $x \in [a,b]$ We say that the *infimum of f on* $[a,b]$ is the greatest lower bound for all values of f (in other words, it is the greatest lower bound of the range of f). We denote the infimum by $\inf_{[a,b]} f$. We say that the *supremum of f on* $[a,b]$ is the least upper bound for all values of f. We denote the supremum by $\sup_{[a,b]} f$.

Exercise 0.9. Give an example of a function $f\colon [a,b] \to \mathbb{R}$ that is not bounded. (*Hint: f* cannot be continuous)

Exercise 0.10. Show that a function $f : [a,b] \to \mathbb{R}$ is bounded if and only if the range of f is a bounded set.

Exercise 0.11. Show that if $f\colon [a,b] \to \mathbb{R}$ is continuous then the maximal value of f is $\sup_{[a,b]} f$ and the minimal value is $\inf_{[a,b]} f$.

Hint: Think back to section 2.7.