EXTRA PROBLEMS #5

DUE: FRI OCTOBER 26TH

In this assignment, we consider again rational and irrational numbers.
We begin with a warm-up exercise (that will also be useful).

Exercise 0.1. Suppose $f : A \to B$ is onto (here the domain of f is A and the codomain is B). Also suppose that $g : B \to C$ is onto. Recall that $g \circ f$ (the composition of the functions g and f) is the function associated to the rule $(g \circ f)(x) = g(f(x))$. Prove that $g \circ f : A \to C$ is also onto.

Exercise 0.2. First show that there exists a function $f \colon \mathbb{N} \to \mathbb{Q}_{>0}$ that is onto. Here $\mathbb{Q}_{>0}$ is used to denote the set of positive rational numbers.

Hint: We are trying to organize the rational numbers into an infinite list. Consider arranging the positive rational numbers like the diagram below.

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & \dots \\ 3/1 & 3/2 & 3/3 & 3/4 & \dots \\ 4/1 & 4/2 & 4/3 & 4/4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

Exercise 0.3. Show that there is an onto function $g : \mathbb{N} \to \mathbb{Q}$.

Hint: First construct an onto function from $\mathbb{Z}$ to $\mathbb{Q}$, then use a previous extra math assignment to find an onto function from $\mathbb{N}$ to $\mathbb{Z}$. Conclude by applying the first exercise.

In Set #3 we showed that every interval contains infinitely many rational numbers. Use the same kind of ideas to

Exercise 0.4. Show that

- Between two rational numbers there is an irrational number.
- Between two irrational numbers there is an rational number.

This exercise can give a feeling that there are as many irrationals than rationals. Since both sets have infinitely many numbers, this is true in some sense. However, sets with infinitely many numbers can be quite different and we will argue in what follows that in some sense there are more irrationals than rationals.

Theorem 0.5 (Cantor)**.** *There is no function $f\colon \mathbb{N} \to [0,1)$ that is onto.*

Our next goal is to prove this theorem.

We will prove the theorem by contradiction. Suppose we have such a function f. Write each $f(n)$ as a decimal expansion and arrange them in a list like so,

$$\begin{array}{lllllllll} f(1) = & 0 & . & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & \dots \\ f(2) = & 0 & . & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \dots \\ f(3) = & 0 & . & d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & \dots \\ f(4) = & 0 & . & d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & \dots \\ \dots & & & & & & & & \end{array}$$

We also assume that each decimal expansion does *NOT* have an infinite trail of 9s at the end. (Remember $0.64999999999\ldots = 0.65$). Recall that once we make this assumption, the decimal expansion of a number is unique (that is, there is only one way to do it).

Exercise 0.6. Prove Theorem 0.5.

*H*int: Draw a diagonal line down and to the right, starting at the digit d_{11}. By considering the digits on that diagonal, explain why you can construct a new decimal number that is

(a) Not on the list.
(b) Inside the interval $(0, 1)$.
(c) Does not have any 9s at all in it (let alone an infinite trail of 9s).

Exercise 0.7. Give an example of an onto function $h : \mathbb{R} \to [0, 1)$.

Exercise 0.8. Prove that there is no onto function $g : \mathbb{N} \to \mathbb{R}$.

*H*int: Suppose there was, then use the previous two exercises and exercise 0.1 to prove this is impossible.

We are now ready to show that there are "more" irrationals than rationals. But first a bit of notation, $\mathbb{R} \setminus \mathbb{Q}$ is just an expression that means *all the real numbers* that are *not rational.* That is $\mathbb{R} \setminus \mathbb{Q}$ is equal to the set of irrational numbers.

Exercise 0.9. Show that there is no function $f\colon \mathbb{Z} \to \mathbb{R} \setminus \mathbb{Q}$ that is onto.

*H*int: Use Theorem 0.5 together with Exercise 0.3 and also the last problem of Set 3.