EXTRA PROBLEMS #4

DUE: FRI OCTOBER 5TH

In this assignment, we'll learn about another way to think about continuity. First we need to define some terms.

1. Definitions

Recall that given a set T (of things, possibly T is a bunch of numbers), a subset U of T is a collection of things inside T. For example, $\{1, 2, 4, 7\}$ is a subset of $\mathbb{N}$. The even numbers are also a subset of $\mathbb{N}$. Furthermore $\mathbb{N}$ is a subset of $\mathbb{Z}$ and $\mathbb{Z}$ is a subset of $\mathbb{Q}$. And finally $\mathbb{Q}$ is a subset of $\mathbb{R}$.

If U is a subset of T, we write $U \subseteq T$. So in the previous example, we have

$$\{1, 2, 4, 7\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Given two subsets U and V of $\mathbb{R}$, we can construct two other subsets using them. The union of two sets U and V, denoted by $U \cup V$ is the collection of all elements of $\mathbb{R}$ that are in either U or V. The intersection of U and V, denoted by $U \cap V$ is all the elements of $\mathbb{R}$ that are in both of U and V. See section 0-4 in the book for additional discussion of these notions.

Hopefully the above was all pretty easy. We begin the real definitions with the notion of the *inverse image of a subset.*

Definition 1.1. Let $f : S \to T$ be a function where S is the domain of f and T is the codomain. Suppose that U is a subset of T (that is, $U \subseteq T$). We define the *inverse image of U under f*, denoted by $f^{-1}(U)$ to be the following subset of S.

$$f^{-1}(U) = \{x \in S | f(x) \in U\}$$

In other words, $f^{-1}(U)$ is all the elements of S that f sends into U.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $f(x) = x^2$. Let U be the interval $(1, 4)$. Then $f^{-1}(U)$ is made up of two intervals $(-2, -1)$ and $(1, 2)$. We can use a union sign to represent $f^{-1}(U) = (-2, -1) \cup (1, 2)$.

Exercise 1.3. Consider $g : (-4\pi, 4\pi) \to \mathbb{R}$ be defined by $g(x) = \sin(x)$. Suppose $U = (0, 1)$. Compute $g^{-1}(U)$.

Finally we define the notion of an open set.

Definition 1.4. We say that a subset $U \subset \mathbb{R}$ is *open* if for every element $c \in U$, there exists a positive real number $d > 0$ such that the interval $(c - d, c + d) \subseteq U$.

Exercise 1.5. Suppose that $U = (a, b)$ is a non-empty open interval. Prove that U is an open set.

Exercise 1.6. Give an example of an open subset of $\mathbb{R}$ that is not an open interval.

Exercise 1.7. Suppose that U and V are two open subsets of $\mathbb{R}$. Prove that $U \cup V$ is an open subset of $\mathbb{R}$. Also prove that $U \cap V$ is an open subset of $\mathbb{R}$.

2. Another characterization of continuity

Our real goal is the following theorem, that you will prove shortly.

Theorem 2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point c of $\mathbb{R}$ if and only if, for every open subset U of $\mathbb{R}$ (thought of as in the codomain), $f^{-1}(U)$ is an open subset of $\mathbb{R}$ (thought of as in the domain).*

Note that in this theorem, there is an "if and only if", which means both conditions are equivalent.

Exercise 2.2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every $c \in \mathbb{R}$, prove that for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an open subset of $\mathbb{R}$.

Exercise 2.3. Suppose that for every open subset $U \subseteq \mathbb{R}$, we have $f^{-1}(U)$ is an open subset of $\mathbb{R}$. Prove that f is continuous.

One reason that we might want this characterization of continuity, is that sometimes we might want to make sense of continuous functions between sets besides simply $\mathbb{R}$. For example, one might want to be able to ask about continuous maps from the surface of a donut shape onto the surface of a sphere.