EXTRA PROBLEMS #3

Due: Fri Sep 28.

In this assignment we consider the rational numbers as a subset of real numbers. Recall, that a number x is rational if we have $x = \frac{m}{n}$ for some integers m and n (where $n \neq 0$, naturally). The first question is a slight modification of a classical problem:

Exercise 0.1. Show that $\sqrt{3}$ is not a rational number.

Hint: The proof can be based on the following fact: every integer $k \geq 2$ can be *UNIQUELY* written as $k = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ positive integers. Primes $p_1, \ldots, p_k$ are called as *prime factors* of k. Based on this fact you notice that every rational number q can be written as $q = m/n$, where m and n are integers, $n \geq 1$, and m and n do not have common prime factors. To prove that $\sqrt{3}$ is not a rational number, assume that it is (say assume $\sqrt{2} = \frac{m}{n}$) and obtain a contradiction using the fact above.

By the solution to Problem 0.1, we know that there are numbers which are not rational, i.e., they are *irrational*. However, at least in case of $\sqrt{3}$, there are rationals as close to $\sqrt{3}$ as you want, since we can show the following:

Exercise 0.2. Let (a, b) be any open interval containing $\sqrt{3}$. Show that there exist rational numbers q and q' so that $a < q < \sqrt{3}$ and $\sqrt{3} < q' < b$.

Hint: There are several ways to do this. One way might be to try considering decimal expansions, but there are less “dirty” ways to do it also.

To see what is meant by “as close as you want” consider very small intervals around $\sqrt{3}$. Now use your solution to Problem 0.2 to show the following:

Exercise 0.3. Let (a, b) be any open interval containing $\sqrt{3}$. Show that there exist infinitely many rational numbers in (a, b).

In fact, even more is true:

Exercise 0.4. Show that all (non-empty) intervals contain infinitely many rational numbers.

Problem 0.4 leads to an idea that there is an abundance of rational numbers. Since all rational numbers are real numbers there are even more reals than rationals. Since all integers are rationals, we could think, having Problem 0.4 in mind, that there are more rationals than integers. This is, however, not true. As an appetizer, consider the following problem.

Exercise 0.5. Show that there exists a function f from $\mathbb{N}$ to $\mathbb{Z}$ so that f is onto.

We return to the subject of the amount of rationals later.