# MATH 185-4, EXAM #1

SOLUTIONS

**1.** Give a precise definition of what it means for two vectors **u** and **v**, in the plane, to be orthonormal. (5 points)

The vectors **u** and **v** are orthonormal if they are both unit vectors and they are orthogonal (aka perpendicular) to each other.

**2.** Give a precise definition of what the following expression means:

$$\lim_{x \to 1.414} f(x) = 2$$

(5 points)

It means that the limit of $f(x)$ as $x$ approaches 1.414 is equal to 2. More precisely, it means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \neq 1.414$ satisfies $c - \delta < x < c + \delta$, then we have $2 - \epsilon < f(x) < 2 + \epsilon$.

**3.** Let $\mathbf{i}$ and $\mathbf{j}$ be the usual orthonormal basis for the plane. Suppose $t$ is a real number and that the vector $\mathbf{v}$ is given by $\mathbf{v} = \mathbf{i} + t\mathbf{j}$.

(a) For what values of $t$ is $\{\mathbf{i}, \mathbf{v}\}$ a basis? Justify your answer. (5 points)

The vectors $\mathbf{i}$ and $\mathbf{v}$ form a basis as long as they are not collinear. But it is easy to see that the only way that $c\mathbf{i} = \mathbf{v} = \mathbf{i} + t\mathbf{j}$ is if $c = 1$ and $t = 0$ (since $\mathbf{i}$ and $\mathbf{j}$ form a basis). Thus for $t \neq 0$, the vectors $\mathbf{i}$ and $\mathbf{v}$ form a basis.

(b) For what values of $t$ is $\{\mathbf{i}, \mathbf{v}\}$ an orthogonal set of vectors? Justify your answer. (5 points)

The only vectors orthogonal to $\mathbf{i}$ are scalar multiples of $\mathbf{j}$. This can be easily seen geometrically or from the law of cosines. Once you have this, it is easy to see that $\mathbf{v}$ is never a scalar multiple of $\mathbf{j}$ because the equation $c\mathbf{j} = \mathbf{v} = \mathbf{i} + t\mathbf{j}$ impossible. (If it was true, you could write $(1)\mathbf{i} + (t - c)\mathbf{j} = 0$ which is impossible since $\mathbf{i}, \mathbf{j}$ is a basis).

**4.** Consider the function defined on $\mathbb{R}$ by the following rule:

$$f(x) = \begin{cases} x-1, & x \geq 1 \\ 3x-3, & x < 1 \end{cases}$$

Use $\epsilon$'s and $\delta$'s to prove that

$$\lim_{x \to 1} f(x) = 0.$$

(10 points)

Let $\epsilon > 0$ and set $\delta = \epsilon/3$. Suppose that $x \neq 1$ satisfies $1 - \delta < x < 1 + \delta$. Then

$$-\epsilon/3 = -\delta < x - 1 < \delta = \epsilon/3$$

There are two cases.

Case 1. If $x < 1$ then by multiplying through by 3 we have $-\epsilon < 3x - 3 < \epsilon$, and so since $3x - 3 = f(x)$ for $x < 1$, we have $-\epsilon < f(x) < \epsilon$ as desired.

Case 2. If $x > 1$, then we still have $-\epsilon/3 < x - 1 < \epsilon/3$, but $-\epsilon < -\epsilon/3$ and $\epsilon/3 < \epsilon$. Thus, we get

$$-\epsilon < x - 1 < \epsilon,$$

and so we are done since $f(x) = x - 1$ for $x > 1$.

**5.** Suppose you are given a function $g$ that is defined and continuous on $[0, 5]$. Suppose that $g(0) = 2$, $g(1) = 10$, and $g(3) = -1$. Prove that $g$ is *NOT* one-to-one. (10 points)

Hint: Draw a picture to help you understand what is going on.

Note that we have the inequality $10 = g(1) > 2 > g(3) = -1$. By the intermediate value theorem, there exists some $x_0$ satisfying $1 < x_0 < 3$ such that $g(x_0) = 2$. But note that $g(0) = 2$ also, and so $g(x_0) = g(0)$ but $x_0 \neq 0$. Thus $g$ is not one-to-one.

**(EC)** Suppose $f$ is a function defined on all of $\mathbb{R}$. Formulate a precise definition for the following notion (2 points)

$$\lim_{x\to\infty} f(x) = \infty.$$

There is more than one correct answer, but here is one. "For every $N > 61$ there exists a $K \in \mathbb{R}$ so that if $x > K$, then $f(x) > N$."

Use your definition to prove the following (2 points)

$$\lim_{x\to\infty} x^2 = \infty$$

Choose an arbitrary $N > 61$. Let $K = \sqrt{N}$ (note that $N$ is never negative). Then suppose that $x > K = \sqrt{N}$. Then by squaring both sides we get $x^2 > N$. Thus we are done.