COMMENTS ON LINEAR INDEPENDENCE

This is just to remind you about the theorem we've been working on (since it is not stated in this form in the book).

Before tackling this, let me say a word about different proof techniques. Let's assume we are trying to prove the following statement. "If condition (A) holds, then condition (B) must hold also". There are three basic approaches to proving this.

(i) The *direct* proof. Simply assume that (A) holds, and then show that (B) must hold also. An example of this would be the proof that (1) implies (2) in the theorem below.

(ii) The proof by *contrapositive*. Assume that (B) is false, and then show that (A) must also be false. An example of this would be the proof that (1) implies (4) in the theorem below.

(iii) The proof by *contradiction*. Assume (A) is true and simultaneously assume that (B) is false. Then show that this is impossible (which is to say, that there is a contradiction). An example of this would be the proof that (3) implies (4) in the theorem below.

Theorem 0.1. *Let* $\mathbf{u}$*,* $\mathbf{v}$ *be a pair of vectors in the plane. Then the the following are equivalent (TFAE).*

(1) *The vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly independent.*

(2) *Every vector* $\mathbf{w}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$

(3) *There exists (∃) a vector* $\mathbf{w}$ *such that the equation*

$$\mathbf{w} = x\mathbf{u} + y\mathbf{v}$$

has exactly one solution in terms of x *and* y*.*

(4) *The equation*

$$\mathbf{0} = x\mathbf{u} + y\mathbf{v}$$

has exactly one solution in terms of x *and* y *(in particular* $x = 0$ *and* $y = 0$ *is that solution).*

(5) *For every vector* $\mathbf{w}$*, the equation*

$$\mathbf{w} = x\mathbf{u} + y\mathbf{v}$$

has exactly one solution, in terms of x *and* y*.*

Proof. (1) implies (2) was already proven in the text (and in class). It's a part of the "basis theorem" in the text. Of course, one should also note that (5) easily implies (2), (3) and (4). We talked about this in class already.

We also showed in class already that (1) is equivalent to (4). To do it, we note that if

$$\mathbf{0} = x\mathbf{u} + y\mathbf{v}$$

is a solution besides $x = y = 0$ (that is, we assumed that (4) was false), then either $x \neq 0$ or $y \neq 0$. If $x \neq 0$, then we can solve the equation for $\mathbf{u}$ and obtain that $\mathbf{u} = \frac{-y}{x}\mathbf{v}$. This means that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ and thus the pair is linearly dependent. If $y \neq 0$ then we can do a similar argument and solve for $\mathbf{v}$. This shows that (1) implies (4).

For the converse (that is, to prove that (4) implies (1)), we assume that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. If $\mathbf{u} = c\mathbf{v}$, then we see that $(1)\mathbf{u} + (-c)\mathbf{v} = \mathbf{0}$ and so we have an alternate solution to $\mathbf{0} = x\mathbf{u} + y\mathbf{v}$, in particular $x = 1$ and $y = -c$. On the other hand, if $\mathbf{v} = c\mathbf{u}$, one can do a similar argument.

Let's now prove that (3) and (4) are equivalent. Note that (4) certainly implies (3), just use $\mathbf{0}$ for $\mathbf{w}$. So all we have to do is prove that (3) implies (4). To do this, we assume that (4) is false and (3) is true and we try to obtain a contradiction (this would imply that (4) must be true). Suppose that

$$\mathbf{0} = x_1\mathbf{u} + y_1\mathbf{v}$$

is a non-trivial solution to the equation (that is, either $x_1 \neq 0$ or $y_1 \neq 0$). Suppose that $\mathbf{w} = x_2\mathbf{u} + y_2\mathbf{v}$ and that this is the only solution. Adding these two equations we get

$$\mathbf{w} = \mathbf{0} + \mathbf{w} = (x_1 + x_2)\mathbf{u} + (y_1 + y_2)\mathbf{v}.$$

Thus we found a second (and distinct) solution to the equation for $\mathbf{w}$. This is a contradiction.

Our next goal is to note that (2) plus (4) implies (5). To prove (5), all we have to do show that every $\mathbf{w}$ has a only one solution in terms of x and y (since we've already assumed (2)). Assume we have two different solutions to the following equation for some $\mathbf{w}$.

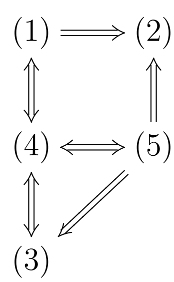
$$\mathbf{w} = x\mathbf{u} + y\mathbf{v}$$

Say the two solutions are x_1, y_1 and x_2, y_2 respectively. Subtracting the two equations with the different solutions plugged in gives us

$$\mathbf{0} = \mathbf{w} - \mathbf{w} = (x_1 - x_2)\mathbf{u} + (y_1 - y_2)\mathbf{v}.$$

Since we started with two different solutions, either $x_1 - x_2 \neq 0$ or $y_1 - y_2 \neq 0$, which contradicts (4). Therefore, whenever (2) and (4) are satisfied, we see that (5) is satisfied. But we can do even better! Since we've already seen that (4) implies (3) which implies (1) which implies (2), we also see that (4) implies (2), so (4) always implies (5).

At this point, let's review what we've done so far.



So all we need to do is show that (2) implies anything else, and we'll be done.

Lets conclude by showing that (2) implies (1). We'll do this by assuming (1) is false and then showing (2) must be false also. But this is easy because if (1) is false, $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, and thus collinear. Choose any non-zero vector $\mathbf{z}$ in some other direction besides that of $\mathbf{u}$ and $\mathbf{v}$. Then, there is no way that $\mathbf{z}$ can be a linear combination of $\mathbf{u}$ and $\mathbf{v}$ because any linear combination of $\mathbf{u}$ and $\mathbf{v}$ are also collinear with $\mathbf{u}$ and $\mathbf{v}$. $\square$

The square box is used to indicate that the proof is over!