Macaulay2 can compute the singular locus. Let’s learn about that. We’ll also learn about the \texttt{FastMinors} package and a bit about how to play around with testing how long things take to run.

**Exercise 1.** Create a singular ring $R$ (ie, $R = \mathbb{Z}/101[x,y,z]/\text{ideal}(x*y-z^2)$). Run 
\begin{verbatim}
singularLocus R
\end{verbatim}
Figure out what it is doing. Discuss with people around you why this might be slow.

**Exercise 2.** Suppose $R = \mathbb{Z}/101[x,y,z]/\text{ideal}(f)$ and $S = \mathbb{Z}/101[u,v,w]/\text{ideal}(g)$ are two homogeneous coordinate rings of (smooth) curves $X$ and $Y$. Use curves with simpler equations, like $f = x^3 + y^3 + z^3$ or $g = x^3 + y^3 + z^3 + x*y*z$.

Recall that $X \times Y$ can be embedded in $\mathbb{P}^8$ via the Segre embedding. On a section ring, if $T = \mathbb{Z}/101[a_0..a_8]$ this is given by a surjection:
\[ T \longrightarrow R \# S \]
which sends $a_0 \mapsto x*u$, $a_0 \mapsto x*v$, etc.

Macaulay2 won’t create the Segre product for you (well, maybe some packages might), let’s do it manually. Note $R \# S$ in our case is a subring of $\mathbb{Z}/101[x,y,z,u,v,w]/\text{ideal}(f,g)$.

Use this and the \texttt{ker} function to give a presentation of the homogeneous coordinate ring of $X \times Y$ inside $\mathbb{P}^8 \cong \text{Proj}T$. You can check your work for example by making sure the $\dim (T/J) = 3 = 1 + \dim X \times Y$.

**Exercise 3.** Now, take the homogeneous coordinate ring of $X \times Y$ (it will look like $T/J$). Try to compute $\text{elapsedTime singularLocus } (R/J)$. You can give up after a little while. Take a look at jacobian $J$ to understand why this is taking so long. Figure out how many determinants Macaulay2 is trying to compute (note $\dim(T/J) = 3 = \dim(X \times Y) + 1$). (Note the singular locus should be the irrelevant ideal, why?)

In general we don’t know a good way to compute all those determinants, or to compute the non-singular locus a different way. One approach to this problem is to try to compute some of the determinants and see if that’s enough.

Try loading the package:
\begin{verbatim}
needsPackage "FastMinors"
\end{verbatim}
Then run the commands
\begin{verbatim}
elapsedTime regularInCodimension(1, T/J)
elapsedTime regularInCodimension(2, T/J) --this one might take a little while (luck?)
\end{verbatim}
What Macaulay2 is doing is choosing different submatrices, computing their determinants, and periodically checking to see if they prove that that $T/J$ is regular in codimension 1 or 2. (In this case if you chose nonsingular curves, $T/J$ is regular except at the origin, and it’s dimension 3, so it
should be regular in codimension 1 and codimension 2, but not 3). You can see what Macaulay2 is doing by running something like:

```plaintext
elapsedTime regularInCodimension(1, T/J, Verbose=>true)
elapsedTime regularInCodimension(2, T/J, Verbose=>true)
```

If you just want to compute the minors directly, try to use the `chooseGoodMinors` function. Since in this case we want $6 \times 6$ submatrices, try:

```plaintext
partialJ = elapsedTime chooseGoodMinors(20, 6, jacobian J);
```
This will choose an ideal of 20 "interesting" $6 \times 6$ minors.

**Exercise 4.** Probably something more useful might be to prove that the singularLocus of $T/J$ really is the irrelevant ideal (up to radical). You can do that manually by adding one minor at a time. First run:

```plaintext
J1 := J;
```
Now run

```plaintext
J1 = chooseGoodMinors(1, 6, jacobian J, J1, Strategy=>StrategyPoints) + J1; dim J1
```
Rerun that line over and over until you get 0. Or make a loop to do it manually. `Strategy=>StrategyPoints` is a very accurate strategy for finding good minors, but finding each minor can be slow (you can also run it on the `regularInCodimension` functions above with Verbose on, to see it in action). When running this we are adding one minor at a time, and checking the dimension.

Once you get zero, you’ve found an ideal of minors that defines a zero dimensional locus. Compute `radical J1` to verify it really is the irrelevant ideal.

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1. determinants of certain submatrices