MATH 5320 - SAMPLE EXAM II

1) Use the Eiseinstein Criterion to prove that $x^6 + x^3 + 1$ is irreducible. (Hint: replace x by $x + 1$.)

2) Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be the map defined by $f(x) \mapsto f(\frac{1}{2} + i)$. Let I be the kernel of φ. Prove that I is principal, i.e. find a generator $g(x)$ and prove that any element in I is a multiple of $g(x)$. Hint: you will need to use Gauss' Lemma for this.

3) Prove that the ring $\mathbb{Z}[\sqrt{-2}]$ is euclidean with respect to the norm $N(x+y\sqrt{-2}) = x^2+2y^2$, i.e. for every $\alpha, \beta \in \mathbb{Z}[\sqrt{-2}]$, with $\beta \neq 0$, show that there exists $\gamma, \delta \in \mathbb{Z}[\sqrt{-2}]$, such that $\alpha = \gamma\beta + \delta$, and $N(\delta) < N(\beta)$. Do this for $\alpha = 4 + 2\sqrt{-2}$ and $\beta = 1 + \sqrt{-2}$.

4) Let $R = \mathbb{Z}[\sqrt{-2}]$. Let p be a prime. When is the principal ideal $(p) \subseteq R$ maximal? (Hint: use $R \cong \mathbb{Z}[x]/(x^2 + 2)$ to understand $R/(p)$.) Use this to determine primes p that can be written as $p = x^2 + 2y^2$. (Using the quadratic reciprocity, the answer depends on p modulo 8).

5) Let R be a ring such that any ideal is finitely generated. Let $I_1 \subseteq I_2 \subseteq I_2 \subseteq \ldots$ be an infinite sequence of ideals in R. Prove that exists an integer n such that $I_n = I_{n+1} = \ldots$.