MATH 5210, HW II
SOLUTIONS

1) A metric space X is separable if it contains a dense countable set S. Prove that any open set V in X is a union of balls centered at points in S and with rational radii. (Since the set of such balls is countable, it follows that any open set is a countable union of balls).

Solution: Let $x \in V$. Then there exists rational $\epsilon > 0$ such that $B(x, \epsilon) \subset V$. Since S is dense, there exists $y \in S$ contained in $B(x, \epsilon/2)$. Clearly x is contained in $B(y, \epsilon/2)$ and this ball is contained in $B(x, \epsilon)$ by the triangle inequality. Hence $B(y, \epsilon/2)$ is contained in V.

2) Let $X = [0, 1]^2$. Choose the distance on X wisely, and use the previous exercise to prove that any open set in X is Lebesgue measurable.

Solution: Let S be the set of points $x = (x_1, x_2)$ in X with both coordinates rational. We let $d(x, y) = \sup(|x_1 - y_1|, |x_2 - y_2|)$. Balls for this choice of distance are rectangles, hence elementary sets, hence measurable. By the previous exercise every open set is a countable union of such rectangles, hence it is measurable, since the set of measurable sets is a σ-algebra.

Remark: With this exercise completed, we at last know that the circle $x_1^2 + x_2^2 < 1$, being an open set, has a well defined area.

3) Let $P = [0, 1]^2$. If E and F are two elementary sets such that $E \cup F = P$ then $m(E \cap F) = m(E) + m(F) - 1$. Now assume $E = \cup_{i=1}^{\infty} E_i$ and $F = \cup_{j=1}^{\infty} F_j$, disjoint unions of elementary sets each, and $E \cup F = P$. Observe that $E \cap F$ is the disjoint union of $E_i \cap F_j$. Prove that

$$\sum_{i,j} m(E_i \cap F_j) = \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

Solution: Fix n, and let $A_n = \cup_{i=1}^{n} E_i$ and $B_n = \cup_{j=1}^{n} F_i$. Since A_n and B_n are elementary sets,

$$m(A_n \cap B_n) = m(A_n) + m(B_n) - m(A_n \cup B_n)$$

Using this inequality, substituting $m(A_n) = \sum_{i=1}^{n} m(E_i)$ and $m(B_n) = \sum_{j=1}^{n} m(F_j)$, we arrive to

$$\sum_{i,j \leq n} m(E_i \cap F_j) = \sum_{i=1}^{n} m(E_i) + \sum_{j=1}^{n} m(F_j) - m(A_n \cup B_n)$$

valid for every n. Let $C_n = A_n \cup B_n$. Observe that C_n is an increasing sequence of elementary sets whose union is P. The problem follows by passing to limit $n \to \infty$ since $\lim_n m(C_n) = 1$: Indeed, we have a disjoint union

$$C_1 \cup (C_2 \setminus C_1) \cup (C_3 \setminus C_2) \cup \ldots = P$$

of elementary sets. It follows (the argument using compactness of P) that $\lim_n m(C_n) = 1$.

4) Let $\sum_{n=1}^{\infty} x_n$ be a series of non-negative real numbers. Show that its sum (which can be ∞) is equal to the supremum of the set of sums $\sum_{n\in S} x_n$ where S runs over all finite subsets of the set of natural numbers. Conclude that any sequence of non-negative numbers can be added in any order.

Solution: Let $S_N = \{1, 2, \ldots, N\}$. By the defintion, $\sum_{n=1}^{\infty} x_n$ is the limit of the sequence of partial sums $\sum_{n\in S_N} x_n$ as $N \to \infty$. Since x_n are non-negative, the sequence of partial sums is monotone increasing, hence $\sum_{n=1}^{\infty} x_n$ is the supremum of the set of finite sums $\sum_{n\in S_N} x_n$. For any finite set S of natural numbers there exists N such that $S \subset S_N$. Then

$$\sum_{n\in S} x_n \leq \sum_{n\in S_N} x_n.$$

Hence the supremum of the set of all finite sums is equal to the supremum of the set of finite sums taken over S_N only. But the former is independent of the ordering of the sequence of real numbers x_n.

5) In the following exercises, $\mathcal{M}$ is a σ-algebra of a non-empty set X, that is, a family of subsets of X closed under complements and countable unions, and μ is a σ-measure. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of sets in $\mathcal{M}$. Let $A = \cap_{i=1}^{\infty} A_i$. Prove that $\lim_{i\mapsto\infty} \mu(A_i) = \mu(A)$, assuming that $\mu(X) = 1$.

Solution: $A^c = \cup_{i=1}^{\infty} A_i^c$, where A^c is the complement of A in X. Since $A_1^c \subseteq A_2^c \subseteq \ldots$ it follows that

$$\lim_{i\mapsto\infty} \mu(A_i^c) = \mu(A^c).$$

Substitute $\mu(A_i^c) = 1 - \mu(A_i)$, $\mu(A^c) = 1 - \mu(A)$, and use elementary properties of limits of sequences.

Observe that the statement fails without assuming the measure of A_1 is finite. Take, for example, $X = \mathbb{R}$ and $A_n = [n, \infty)$ then $\mu(A_n) = \infty$, for all n, $A = \emptyset$ and $\mu(A) = 0$.

6) A subset of X is called measurable if it belongs to $\mathcal{M}$. Let $f : X \to \mathbb{R}$ prove that

$$\{x | f(x) < c\}$$

is measurable for every $c \in \mathbb{R}$ if and only if

$$\{x | f(x) \leq c\}$$

is measurable for every $c \in \mathbb{R}$.

Solution: Equivalence of the two follows from the following set-theoretic identities:

$$\{x | f(x) < c\} = \cup_n \{x | f(x) \leq c - \frac{1}{n}\}$$

and

$$\{x | f(x) \leq c\} = \cap_n \{x | f(x) < c + \frac{1}{n}\}$$

7) Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions on X. Prove that

$$g(x) = \inf\{f_1(x), f_2(x), \ldots\} \text{ and } G(x) = \sup\{f_1(x), f_2(x), \ldots\}$$

are measurable functions.

Solution:

$$\{x|g(x) < c\} = \cup_n \{x|f_i n(x) < c\}.$$
$$\{x|G(x) \leq c\} = \cap_n \{x|f_n(x) \leq c\}.$$

Now use the previous exercise.

8) Let f be an integrable function on X, such that $f(x) \geq 0$ for all $x \in X$. Prove that $\int_X f = 0$ if and only if the measure of $A = \{x \in X \mid f(x) > 0\}$ is 0, that is, $f = 0$ almost everywhere. Hint consider the sets $A_n = \{x \in X \mid f(x) > 1/n\}$ for $n = 1, 2, \ldots$.

Solution: Assume that $\int f_X = 0$. Let χ_n be the characteristic function of A_n multiplied by $1/n$. It is a simple function whose integral is $\mu(A_n)/n$. Since

$$0 = \int_X 0 \leq \int_X \chi_n \leq \int_X f = 0$$

it follows that $\mu(A_n) = 0$. Now observe that $A_1 \subseteq A_2 \subseteq \ldots$ and A is the union of A_n. Hence $\mu(A) = \lim_n \mu(A_n) = 0$.

In the other direction, for every n, let $f_n = \sum_{m=1}^{\infty} m \cdot \chi_{X_m}$ be the simple function where

$$X_m = \{x \in X | \frac{m-1}{n} < f(x) \leq \frac{m}{n}\}$$

Observe that $\mu(X_m) = 0$ if $m > 0$. Hence $\int_X f_n = 0$. Moreover, f_n converges uniformly to f, hence $\int_X f = 0$ from the definition of the integral. Observe that this argument gives a bit more: a measurable function f equal 0 almost everywhere is integrable and its integral is 0.

9) Let $X = (0, 1]$, with the usual measure, and let $f(x) = 1/\sqrt{x}$. Use the monotone convergence theorem to prove that f is integrable and compute its integral.

Solution: Let f_n be the product of f and the characteristic function of $[\frac{1}{n}, 1]$. Then f_n is a monotone sequence with f point-wise limit. Since f_n is continuous on $[\frac{1}{n}, 1]$, its Lebesgue integral is equal to the Riemann integral which we can compute using the Fundamental Theorem of Calculus:

$$\int_{1/n}^{1} \frac{1}{\sqrt{x}} dx = 2(1 - \frac{1}{\sqrt{n}}) < 2.$$

By the MCT f is integrable and

$$\int_X f = \lim_n \int_X f_n = 2.$$