MATH 5210, HW I
SOLUTIONS

1) In this and the following problem, use $d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$ as the distance function on $\mathbb{R}^2$. Use ϵ-δ definition of continuity to prove that the multiplication map $\mathbb{R}^2 \to \mathbb{R}$ is continuous.

Solution: Given $\epsilon > 0$, we need to show that there is $\delta > 0$ such that $|x_1x_2 - y_1y_2| < \epsilon$ if $d(x,y) < \delta$. We can ssume that x and y are contained in a large square $[-M, M]^2$. Then

$$|x_1x_2 - y_1y_2| = |x_1x_2 - y_1x_2 + y_1x_2 - y_1y_2| < |x_2||x_1 - y_1| + |y_1||x_2 - y_2| < 2M\delta$$

so we can take $\delta = \epsilon/2M$.

2) Let $p_i : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the i-th coordinate. Prove that p_i is continuous. Let (X, d) be a metric space. Let $f : X \to \mathbb{R}^2$ be a map, and write $f(x) = (f_1(x), f_2(x))$ for every $x \in X$. In particular we have two functions $f_i : X \to \mathbb{R}$, $i = 1, 2$. Prove that f is continuous if and only if f_1 and f_2 are.

Solution: If $d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|) < \epsilon$ then $|p_i(x) - p_i(y)| = |x_i - y_i| < \epsilon$, hence the projection maps are uniformly continuous.

Observe that $f_i = p_i \circ f$. If f is continuous, then f_i is continuous, being a composite of two continuous maps. Now assume that f_1 and f_2 are continuous. Observe that

$$f^{-1}((a,b) \times (c,d)) = f_1^{-1}((a,b)) \cap f_2((c,d)).$$

This set is open, since $f_1^{-1}((a,b))$ and $f_2^{-1}((c,d))$ are open, by continuity. Hence f is continuous.

3) Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$. Use the inductive definition $x^n = x \cdot x^{n-1}$ and previous exercises to prove that f is continuous.

Solution: By induction. Assume that $x \mapsto x^{n-1}$ is continuous. Then $x \mapsto x^n$ is a composite of two maps

$$x \mapsto (x, x^{n-1}) \mapsto x \cdot x^{n-1}$$

where the first is continous by exercise 2) and the second by exercise 1).

4) Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$. Prove that

$$\int_a^b f = 0$$

implies $f(x) = 0$ for all $x \in [a,b]$.

Solution: Assume $f \neq 0$, we would like to show that the integral of f is positive. It suffices to find one positive lower sum. Let $c \in (a,b)$ such that $f(c) > 0$. Let $\epsilon = f(c)/2$. Since f is continuous, there exists $\delta > 0$ such that $f(x) > \epsilon$ if $|x - c| < \delta$. Take the partition of $[a, b]$ that includes $[c - \delta, c + \delta]$ as a subsegment. The corresponding lower sum is greater than $2\delta\epsilon$.

5) Let (X, d) be a metric space. Let (x_n) and (y_n) be two Cauchy sequences in X. Prove that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$.

Solution: By the triangle inequality,

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and this implies that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_n, y_m)$$

By switching the roles of n and m we also get that

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_n, y_m)$$

The two are equivalent to

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_n, x_m) + d(y_n, y_m)$$

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, for every $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \epsilon/2$ and $d(y_n, y_m) < \epsilon/2$ for all $n, m \geq N$. Hence

$$|d(x_m, y_m) - d(x_n, y_n)| < \epsilon$$

if $n, m \geq N$.

6) Let $K \subset \mathbb{R}$ be a set consisting of 0 and all $1/n$, $n = 1, 2, 3, \ldots$. Prove that K is compact directly using the definition, i.e. every open cover has a finite subcover.

Solution: Let O_α, $\alpha \in S$, be an open covering of K. Then there exists $\beta \in S$ such that $0 \in O_\beta$. Since O_β is open, there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset O_\beta$. Let N be an integer such that $1/N < \epsilon$. Then $1/n \in O_\beta$ for all $n \geq N$. It follows that K is covered by O_β and finitely many O_α needed to cover $1/n$, for $n < N$.

7) Let $F_1 \supseteq F_2 \supseteq \ldots$ be a descending sequence of non-empty compact subsets. Prove that $\cap_{n=1}^{\infty} F_n$ is non-empty.

First solution: Pick $x_n \in F_n$. Since F_1 is compact, a subsequence of x_n converges to a point $x \in F_1$. But x is in all F_n, since they are closed.

Second solution: If $\cap_{n=1}^{\infty} F_n$ is empty, then $F_1 \setminus F_n$ is an open cover of F_1 that cannot be reduced to a finite subcover, a contradiction.

8) Let (X, d) be a metric space and f_n a sequence of continuous functions $f_n : X \to \mathbb{R}$ uniformly converging to f. Let x_n be a sequence of points in X such that $\lim_n x_n = x \in X$. Prove that $\lim_n f_n(x_n) = f(x)$.

Solution: Let $\epsilon > 0$. The function f is continuous, so there exists N_1 such that $|f(x) - f(x_n)| < \epsilon/2$ for all $n > N_1$. The sequence converges uniformly to f, so there exists N_2 such that $|f(x) - f_n(x)| < \epsilon/2$ for all $n > N_2$ and all x. Let N be the grater of N_1 and N_2. If $n > N$ then

$$|f(x) - f_n(x_n)| \leq |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

9) A subset $\mathbb{R}^n$ is convex if for any two points $x, y \in C$, the segment $[x, y]$ is contained in C. Prove that C is connected.

Solution: A point of this exercise is understand how proofs are built on previous proofs. Let E be a convex set. Assume that $E = A \cup B$ where A and B are two non-empty separating sets. Let $a \in A$ and $b \in B$. Let $[a, b]$ be the straight segment connecting a and b. Since E is convex the whole segment is contained in E. Hence $A \cap [a, b]$ and $B \cap [a, b]$ are separating sets for $[a, b]$. (A limit point of $A \cap [a, b]$ is also a limit point of A, hence it is not contained in B, since $\bar{A} \cap B = \emptyset$, and therefore not in $B \cap [a, b]$.) But we proved that the straight segments are connected.