HW IV FOR MATH 6210

1) Let H be a Hilbert space and $P : H \to H$ a continuous operator such that $P^2 = P$ and $P^* = P$. Prove that the image of P is closed and that H is isomorphic to the direct sum of the kernel of P and the image of P.

2) Let H be a Hilbert space and e_1, $e_2, \ldots$ and an orthonormal basis. Prove that 0 is a weak limit of the sequence e_n. Prove that the unit ball B, $||x|| \leq 1$, is closed in the weak topology. Prove that the unit sphere S, $||x|| = 1$, is dense in B in the weak topology.

3) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two σ-finite measure spaces. If functions $e_1, e_2 \ldots$ are a Hilbert space basis of $L^2(X)$ and functions $f_1, f_2, \ldots$ are a Hilbert space basis of $L^2(Y)$, prove that $e_i \cdot f_j$ is a basis of $L^2(X \times Y)$. (Here $e_i \cdot f_j$ is a function on $X \times Y$ such that its value at $(x, y) \in X \times Y$ is $e_i(x) \cdot f_j(y)$.)

4) Let $X = [0, 1]$ with the usual measure. Let $p \geq 1$. For $n = 1, 2, \ldots$, let $f_n = n^{1/p} \cdot \chi_{[0,1/n]}$. Prove that the sequence f_n converges weakly to 0 in $L^p([0, 1)]$ if and only if $p > 1$. (Use that $L^p([0, 1)]^* \cong L^q([0, 1)]$.)

5) Let $1 < p < q < \infty$. Then L^p is not contained in L^q iff X contains sets of arbitrarily small positive measure, and L^q is not contained in L^p iff X contains sets of arbitrarily large finite measure. This is exercise 5, page 187, see the hints there.

6) Exercise 18 on page 191. Follow the hints there to prove the Radon-Nikodym Theorem for a finite measure space $(X, \mathcal{M}, \mu)$, using $L^2(X)^* \cong L^2(X)$ which is a part of the Hilbert space theory. (As usually, for simplicity assume we deal with real valued functions.)