HW III FOR MATH 6210

1) Let V be a normed space and U a closed subspace. For every $x \in V$, let

$$||x+U|| = \inf\{||x+y|| \mid y \in U\}.$$

(1) Prove that $||x+U||$ is a norm on V/U.
(2) For every $\epsilon > 0$ there exists $x \in V$ such that $||x|| = 1$ and $||x+U|| \geq 1-\epsilon$.
(3) If V is complete, so is V/U.
(4) Let $P : V \to V/U$ be $P(x) = x+U$. Prove that P has norm 1.

2) Let $T : V \to W$ be a bounded map. Let U be the kernel of T.

(1) Prove that U is closed.
(2) Prove that $||S|| = ||T||$ where $S : V/U \to W$ such that $S(x+U) = T(x)$, for all $x \in V$.

3) Let $T : V \to W$ be a map between two normed spaces, where W is finite dimensional. Let U be the kernel of T. Prove that T is bounded if U is closed.

4) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E \in \mathcal{M}$ of finite and positive measure. Let $T : L^1(X) \to \mathbb{R}$ be defined by $T(f) = \int_E f$. Prove that T is a bounded functional, and compute its norm.

5) Let $T : V \to U$ be a bounded map between two normed spaces. Let $T^* : U^* \to V^*$ be defined by $T^*(f) = f \circ T$ for all $f \in U^*$ (the adjoint map).

(1) Prove that $||T^*|| = ||T||$.
(2) If we identify V and U with their canonical images in V^{**} and U^{**} prove that the restriction of T^{**} to V coincides with T.

6) Let $V = C([0,1])$ be the space of continuous functions equipped with the sup norm. Let $U = C^1([0,1])$ be the subspace of continuously differentiable functions. Let $T : U \to V$ be defined by $T(f) = f'$. Prove that the graph of T is closed, and that T is not bounded.

7) Let $T : V \to U$ be a linear map between two Banach spaces such that if f is a continuous functional on U, then $f \circ T$ is a continuous functional on V. Prove that T is bounded. Hint: prove that the graph of T is closed, to that end use Theorem 5.8 c on page 159.