HW III FOR MATH 6210
DUE MONDAY OCTOBER 09

1) Please justify all steps in the following:

(1) Prove that $f(x) = x\exp(-\frac{x^2}{2})$ is in $L^1(\mathbb{R})$.

(2) Use Theorem 2.27 to compute the derivative of $g(t) = \int_{\mathbb{R}} \exp(-\frac{x^2}{2})\cos(tx)dx$, where $t \in \mathbb{R}$.

(3) Use $\exp(-\frac{x^2}{2})' = -x\exp(-\frac{x^2}{2})$ and integration by parts to show that $g(t)$ satisfies a first order differential equation. Solve it to find g.

2) Let $\mathcal{M} \subseteq \mathcal{P}(X)$ be the σ algebra generated by an elementary family $\mathcal{E}$, see the definition on page 23. Let μ and ν be two positive finite measures on $\mathcal{M}$, such that $\mu(E) = \nu(E)$ for all E in $\mathcal{E}$. Prove that $\mu = \nu$.

In the following exercises μ denotes the standard Lebesgue measure on the Borel algebra of $\mathbb{R}^n$. Let P^n be the open punctured ball $0 < ||x|| < 1$ and S^{n-1} the sphere $||x|| = 1$, where $||x||$ is the standard norm on $\mathbb{R}^n$. Note that P^n is homeomorphic to $(0,1) \times S^{n-1}$. Let dr denote the standard Lebesgue measure on $(0,1)$.

3) Take $n = 2$ above. Let $d\varphi$ be the Lebesgue measure on the unit circle S^1 (the arc length). Let $dr\ d\varphi$ be the product measure on the Borel algebra of $(0,1) \times S^1 \cong P^2$. Prove that for every Borel set E in P^2,

$$\mu(E) = \int_E r dr\ d\varphi.$$

4) Let F be a Borel set in S^{n-1}. Then $c(F) = (0,1) \times F$ is a Borel set in P^n. (Why?) Let $\nu(F) = \frac{1}{n}\mu(c(F))$. Prove that ν is a Borel measure on S^{n-1} invariant under the orthogonal transformations.

5) Let ν be the Borel measure on S^{n-1} as in 4). Let $dr \otimes \nu$ denote the product measure on the Borel algebra of $(0,1) \times S^{n-1} \cong P^n$. Prove that for every Borel set E in P^n,

$$\mu(E) = \int_E r^{n-1} dr \otimes \nu.$$

Hint: It suffices to do this for $E = (0,a) \times F$, where F is a Borel set in S^{n-1} (why?) and in this case it reduces to proving that $\mu((0,a) \times F) = a^n \mu((0,1) \times F)$. Why is the last identity true?

The problem 5) combined with 14) on page 52 (the previous HW) gives us the formula for integration in "polar" coordinates in $\mathbb{R}^n$, first on any ball of a finite radius n and then by monotone convergence in general.