# HW I FOR MATH 6210
# DUE MONDAY SEPTEMBER 04

1) Let $X = \mathbb{Z}$, the ring of integers, and fix a prime $p$. For every $x \in \mathbb{Z}$ and integer $n \geq 0$, let

$$B_{x,n} = \{y \in \mathbb{Z} \mid y \equiv x \pmod{p^n}\}.$$

Let $\mathcal{E} \subset \mathcal{P}(X)$ consist of all disjoint unions of $B_{x,n}$, including the empty set. Prove that $\mathcal{E}$ is an algebra, and that $\mu(B_{x,n}) = 1/p^n$ defines a finitely additive measure on it.

2) Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ be two algebras. Let $\mathcal{C} \subseteq \mathcal{P}(X \times Y)$ consist of all disjoint unions of products $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $\mathcal{C}$ is an algebra.

3) Let $\mathcal{A}$ be an algebra of sets, with a finitely additive measure $\mu$. For any two $A, B \in \mathcal{A}$ define $d(A, B) = \mu(A \Delta B)$. Prove that $d$ satisfies the triangle inequality i.e.

$$d(A, C) \leq d(A, B) + d(B, C)$$

for any three sets in $\mathcal{A}$.

4) Let $X = [0, 1) \times [0, 1)$, and $\mu^*$ the Lebesgue outer measure on $X$. Let $A \subseteq X$. If for every $\epsilon > 0$ there exists an elementary set $E$ such that $\mu^*(A \Delta E) < \epsilon$, prove that $\mu^*(A) + \mu^*(A^c) = 1$.

5) Let $\mu^*$ be an outer measure on $X$. Let $A \subseteq X$ be a measurable set (as defined on page 29 in Folland) $\mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E)$ for all sets $E$ in $X$. If $B$ is a subset of $A^c$ prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

6) Let $\mathcal{A}$ be a $\sigma$-algebra and $\mu$ a measure on $\mathcal{A}$. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of sets in $\mathcal{A}$ such that $\mu(A_1)$ is finite. Let $A = \cap_{i=1}^{\infty} A_i$. Prove that $\lim_{i \mapsto \infty} \mu(A_i) = \mu(A)$.

7) Let $\mu$ be a finite Borel measure on $\mathbb{R}$. Let $F(x) = \mu(-\infty, x]$. Show that $F(x)$ is continuous at $x$ if and only if $\mu(\{x\}) = 0$.