1) In this and the following problem, use \(d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)\) as the distance function on \(\mathbb{R}^2\). Use \(\epsilon-\delta\) definition of continuity to prove that the multiplication map \(\mathbb{R}^2 \rightarrow \mathbb{R}\) is continuous.

2) Let \(p_i : \mathbb{R}^2 \rightarrow \mathbb{R}\) be the projection on the \(i\)-th coordinate. Prove that \(p_i\) is continuous. Let \((X, d)\) be a metric space. Let \(f : X \rightarrow \mathbb{R}^2\) be a map, and write \(f(x) = (f_1(x), f_2(x))\) for every \(x \in X\). In particular we have two functions \(f_i : X \rightarrow \mathbb{R}, i = 1, 2\). Prove that \(f\) is continuous if and only if \(f_1\) and \(f_2\) are.

3) Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) given by \(f(x) = x^n\). Use the inductive definition \(x^n = x \cdot x^{n-1}\) and previous exercises to prove that \(f\) is continuous.

4) Let \(f : [a, b] \rightarrow \mathbb{R}\) be a continuous function such that \(f(x) \geq 0\) for all \(x \in [a, b]\). Prove that \[
\int_a^b f = 0
\]
implies \(f(x) = 0\) for all \(x \in [a, b]\).

5) Let \((X, d)\) be a metric space. Let \((x_n)\) and \((y_n)\) be two Cauchy sequences in \(X\). Prove that \((d(x_n, y_n))\) is a Cauchy sequence in \(\mathbb{R}\).

6) Let \(K \subset \mathbb{R}\) be a set consisting of 0 and all \(1/n, n = 1, 2, 3, \ldots\). Prove that \(K\) is compact directly using the definition, i.e. every open cover has a finite subcover.

7) Let \(F_1 \supseteq F_2 \supseteq \ldots\) be a descending sequence of non-empty compact subsets. Prove that \(\bigcap_{n=1}^{\infty} F_n\) is non-empty.

8) Let \((X, d)\) be a metric space and \(f_n\) a sequence of continuous functions \(f_n : X \rightarrow \mathbb{R}\) uniformly converging to \(f\). Let \(x_n\) be a sequence of points in \(X\) such that \(\lim_n x_n = x \in X\). Prove that \(\lim_n f_n(x_n) = f(x)\).

9) A subset \(\mathbb{R}^n\) is convex if for any two points \(x, y \in C\), the segment \([x, y]\) is contained in \(C\). Prove that \(C\) is connected.