MATH 3210-4, HW III
DUE WEDNESDAY OCT 15

1) Let x_n be a sequence of positive real numbers such that $\lim_n x_n = x > 0$. Prove that

- $\lim_n x_n^2 = x^2$.
- $\lim_n \sqrt{x_n} = \sqrt{x}$.

2) Define a sequence by $s_1 = 1$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$. Prove that $s_n < 2$ for all n, and that s_n is an increasing sequence. Hint: prove that $s_{n-1} < s_n$ implies that $s_n < s_{n+1}$. Find the limit.

3) Let $X = \mathbb{Z}$ and $d(x, x) = 0$ or $d(x, y) = \frac{1}{2^n}$, if $x \neq y$, where 2^n is the largest power of 2 dividing $x - y$. Prove that the following two series are Cauchy. One of them is convergent (find its sum) while the other not (explain).

- $\sum_{n=0}^{\infty} 2^n$
- $\sum_{n=0}^{\infty} (-2)^n$

4) Let c_n be a sequence of positive numbers. Prove that

$$\text{liminf}_n \frac{c_{n+1}}{c_n} \leq \text{liminf}_n \sqrt[n]{c_n}.$$

5) Let Y be a non-empty set. A function $d : Y \times Y \to [0, \infty)$ is called a quasi-distance if

- $d(x, x) = 0$ for all $x \in Y$.
- $d(x, y) = d(y, x)$ for all $x, y \in Y$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in Y$.

In words, d is almost a distance function, however, $d(x, y) = 0$ is allowed for different x and y. We say that x and y are equivalent if $d(x, y) = 0$. Prove that this is an equivalence relation. Prove that, if x is equivalent to y then $d(x, z) = d(y, z)$ for all $z \in Y$.

6) Let (X, d) be a metric space. Let $x = \{x_n\}$ and $y = \{y_n\}$ be two Cauchy sequences. Prove that the sequence of distances $d(x_n, y_n)$ is a Cauchy sequence of real numbers.

7) Let (X, d) be a metric space. Let Y be the set of all Cauchy sequence. For any two Cauchy sequences $x = \{x_n\}$ and $y = \{y_n\}$ let

$$d(x, y) = \lim_n d(x_n, y_n).$$

Prove that d is a quasi-distance on Y. (So when we form equivalence classes as in 5), we get a new metric space, called the completion of X.)