MATH 3210-4, HW II, SOLUTIONS

1) A complex number z is called algebraic if there exists integers $a_0, a_1, \ldots, a_n$, such that $a_n z^n + \cdots + a_1 z + a_0 = 0$. Prove that the set of algebraic numbers is countable.

Observe that all rational numbers are algebraic numbers, hence the set of algebraic numbers is certainly not finite. Let the positive integer $|a_0|+2|a_1|+\ldots+(n+1)|a_n|$ be called the height of the polynomial $a_n z^n + \cdots + a_1 z + a_0$. Note that there are only finitely many polynomials of a fixed height, and there are only finitely many algebraic numbers that are their roots. The exercise follows since a countable union of finite sets is countable.

2) Prove that the following two (X, d) are metric spaces:

(1) $X = \mathbb{R}^2$ and $d((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|)$.
(2) $X = \mathbb{Z}$ and $d(x, x) = 0$ or $d(x, y) = \frac{1}{2^n}$, if $x \neq y$, where 2^n is the largest power of 2 dividing $x - y$.

Solution: Proofs of the triangle inequality in each case. (1) Want to prove $d(x, z) \leq d(x, y) + d(y, z)$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Let i be such that $d(x, z) = |x_i - z_i|$. Then

$$d(x, z) = |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d(x, y) + d(y, z).$$

(2) Write $x - y = 2^n r$ and $y - z = 2^m s$ where r and s are odd. Then $x - z = 2^n r + 2^m s$ so $x - z$ is divisible by 2^n or 2^m (whichever is smaller). Hence $d(x, z)$ is smaller than $1/2^n$ or $1/2^m$. In ether case $d(x, z)$ is smaller than

$$d(x, y) + d(y, z) = 1/2^n + 1/2^m.$$

3) Let (X, d) be a metric space. The closed ball centered at x and of radius $r > 0$ is the set of $y \in X$ such that $d(x, y) \leq r$. Prove that the complement of the closed ball is an open set in X.

Solution: Let z be not in the closed ball. Then $d(x, z) = s > r$. Let $\epsilon = s - r$. Claim: The open ball $B(z, \epsilon)$ is contained in the complement of the closed ball. If not, then there is y in the intersection, hence $d(x, y) \leq r$ and $d(y, z) < \epsilon$. By the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z) < r + \epsilon = s$$

a contradiction, since $d(x, y) = s$.

4) Let (X, d) be a metric space. Recall that the closure of $E \subset X$ is $\bar{E} \supseteq E$ obtained by adding limit points to E. Since $\bar{E}$ is larger than E, it seems possible that $\bar{E}$ has additional limit points, i.e. the closure does not give a closed set. Prove that

$$\bar{E} = \cap_{E \subseteq F, \bar{F} = F} F$$

i.e. the intersection is taken over all closed sets F containing E. In particular, $\bar{E}$ is closed, why?

Solution: Generally, in order to prove identity of two sets, we need to prove inclusions in both directions.
(1) $\bar{E} \subset \cap_{E\subseteq F, \bar{F}=F} F$. To prove this inclusion, we need to prove that $\bar{E}$ is contained in any closed set F containing E. That is, we need to prove that any limit point of E is contained in F. But any limit point of E is also a limit point of F (since F contains E). Since F is closed, it contains all its limit points, hence $\bar{E} \subset F$.
(2) $\bar{E} \supset \cap_{E\subseteq F, \bar{F}=F} F$. Let $x \notin \bar{E}$. Then there exists a ball $B(x,r)$ disjoint to $\bar{E}$. Then $E_x := X \setminus B(x,r)$ is a closed set containing $\bar{E}$ and not containing x. It is clear that the intersection of all such E_x is contained in $\bar{E}$ and this proofs the opposite inclusion. (Why?)
As to the last part, the intersection of any collection of closed sets is closed, hence $\bar{E}$ is closed.

Let X be a metric or, more generally, a topological space. A collection of open sets $\{V_\alpha\}$ in X is called a base for X if for any open set V and $x \in V$ there exists a V_α in the collection such that V_α is contained in V and it contains x. In particular, any open set can be written as a union of a subcollection of $\{V_\alpha\}$. For example, if X is a metric space, then the collection of all open balls is a base for X, by the definition of open sets. Topological spaces with a countable base are called separable.

5) Assume that a metric space X contains a countable subset X_0 such that the closure of X_0 is X. Prove that the collection of balls centered at $x \in X_0$ and rational radii is a countable base for X.

Solution: The easy part is that the collection of balls $B(x_0, r)$ where $x_0 \in X_0$ and r rational is countable. Here is a proof that it is a base. So let V be an open set and $x \in V$. We need to prove that there is a ball $B(x_0, r)$ where $x_0 \in X_0$ and r rational, containing x and contained in V. Since V is open, there exists a rational number r such that $B(x,r)$ is contained in V. If $x \in X_0$ we are done. If x is not in X_0 we better use that x is a limit point of X_0, as there is nothing else given to us. , there exists $x_0 \in X_0$ in $B(x, r/2)$, since x is a limit point of X_0. Clearly x is contained in $B(x_0, r/2)$ by symmetry of the distance function. By the triangle inequality (make a drawing) the ball $B(x_0, r/2)$ is contained in $B(x,r)$, and hence it is in V.

6) Prove that convex sets in $\mathbb{R}^2$ are connected, in the sense of definition 2.45.

Solution: A point of this exercise is understand how proofs are built on previous proofs. Let E be a convex set. Assume that $E = A \cup B$ where A and B are two non-empty separating sets. Let $a \in A$ and $b \in B$. Let $[a,b]$ be the straight segment connecting a and b. Since E is convex the whole segment is contained in E. Hence $A \cap [a,b]$ and $B \cap [a,b]$ are separating sets for $[a,b]$. (A limit point of $A \cap [a,b]$ is also a limit point of A, hence it is not contained in B, since $\bar{A} \cap B = \emptyset$, and therefore not in $B \cap [a,b]$.) But we proved that the straight segments are connected.

True, we only proved that the segment $[0,1]$ on the x-axis is connected, but any closed segment in $\mathbb{R}^2$ can be rotated, translated and rescaled onto $[0,1]$. Rotations, translations, and rescaling by a non-zero number clearly permute open balls, hence open and closed sets, map separating sets to separating sets etc...