In order to define multiplication of real numbers, it suffices to do so for positive numbers, and here it is convenient to use cuts of positive rational numbers. Thus a positive real number is defined a non-empty, bounded set \( \alpha \subseteq \mathbb{Q}^+ \) such that

- If \( r \in \alpha \) and \( s < r \) is a positive rational number, then \( s \in \alpha \).
- \( \alpha \) has no maximal element.

An example is “the cut of square root of 3”:

\[ \sqrt{3} = \{ r \in \mathbb{Q}^+ | r^2 < 3 \} \]

The first bullet is clearly satisfied, for the second, note that the limit of \( (r + \frac{1}{n}) \) is \( r^2 \), as the positive integer \( n \) tends to infinity, hence for \( n \) large enough \( (r + \frac{1}{n})^2 < 3 \) if \( r^2 < 3 \).

1) For \( i = 0, \ldots, 10 \), construct the greatest rational number in the cut of square root of 3 in the form of a (non-reduced) fraction \( a_i = x_i/2^i \), and the least rational number not in the cut in the same form \( b_i = y_i/2^i \). For example, \( a_0 = 1 \) and \( b_0 = 2 \). Their average is \( 3/2 \). Since \((3/2)^2 < 3\), it follows that \( a_1 = 3/2 \) and \( b_1 = 4/2 \) etc... Note: if you had a calculator that “computes” \( \sqrt{3} \) and expresses the answer in binary digits, \( a_i \) would be what you get after chopping off all digits after the \( i \)-th place right of the point.

2) Let \( \alpha \) and \( \beta \) be two cuts of \( \mathbb{Q}^+ \). Let \( \alpha \cdot \beta = \{ rs | r \in \alpha, s \in \beta \} \). Prove that \( \alpha \cdot \beta \) is a cut of \( \mathbb{Q}^+ \).

3) Prove that \( \sqrt{3} \cdot \sqrt{3} = 3^* \), where \( \sqrt{3} \) is the cut defined above.

4) Let \( 1^* = \{ r \in \mathbb{Q}^+ | r < 1 \} \). Prove that \( \alpha \cdot 1^* = \alpha \) for any cut \( \alpha \) of \( \mathbb{Q}^+ \).

5) Let \( \alpha \) be a cut of \( \mathbb{Q}^+ \). Construct a cut \( \beta \) such that \( \alpha \cdot \beta = 1^* \).

6) The set of \( 2 \times 2 \) matrices with real coefficients is a non-commutative ring with respect to the usual addition and multiplication of matrices. We can use this information to quickly construct complex numbers and prove that it is a field. Let \( \mathbb{C} \) be the set of \( 2 \times 2 \) matrices

\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \]

where \( a \) and \( b \) are any real numbers. If \( A, B \in \mathbb{C} \), prove that \( A + B \), \( A - B \) and \( AB \) are in \( \mathbb{C} \) and that \( AB = BA \). This implies that \( \mathbb{C} \) is a ring (why?). Finally, for every non-zero \( A \in \mathbb{C} \) find \( B \in \mathbb{C} \) such that \( AB = 1 \) i.e. \( \mathbb{C} \) is a field. What fails if we instead consider \( \mathbb{C}' \), the set of all \( 2 \times 2 \) matrices

\[ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \]

where \( a \) and \( b \) are any real numbers?