Let $V$ be a Euclidean vector space, that is, a vector space over $\mathbb{R}$ with a scalar product $(x, y)$. Then $V$ is a normed space with the norm $||x||^2 = (x, x)$. We shall need the following continuity of the dot product.

**Exercise.** Let $x, y \in V$ and $(x_n)$ a sequence in $V$ converging to $x$. Then

$$\lim_{n} (x_n, y) = (x, y).$$

**Hint:** Use Cauchy Schwarz inequality.

**Solution.** $(x_n)$ converging to $x$ means $\lim_{n} ||x_n - x|| = 0$.

$$||x_n - x|| \cdot ||y|| \geq |(x_n, y) - (x, y)| = |(x_n - x, y)| \leq ||x_n - x|| \cdot ||y||$$

hence $(x_n, y)$ converges to $(x, y)$.

Now assume that $V$ is a Hilbert space, i.e. a separable and complete Euclidean space. Let $e_1, e_2, \ldots$ its orthonormal basis, see the previous lecture. In particular, the subspace $U$ spanned by $e_1, e_2, \ldots$ is a dense subset.

**Lemma 0.1. Bessel's inequality.** For every $v \in V$, and every $n \in \mathbb{N}$,

$$(v, e_1)^2 + \ldots + (v, e_n)^2 \leq ||v||^2.$$  

**Proof.** Let

$$v_n = (v, e_1)e_1 + \ldots + (v, e_n)e_n.$$  

Then, for every $i \leq n$,

$$(v - v_n, e_i) = (v, e_i) - (v_n, e_i) = 0.$$  

Since $v_n$ is a linear combination of $e_i$ for $i \leq n$, it follows that $v_n$ and $v - v_n$ are perpendicular. By the Pythagorean equality,

$$||v_n||^2 \leq ||v||^2 + ||v - v_n||^2 = ||v||^2.$$  

The lemma follows since $||v_n||^2 = (v, e_1)^2 + \ldots + (v, e_n)^2$.  

Now we can prove the main result in the theory of (infinite dimensional) Hilbert spaces.

**Theorem 0.2. (Riesz-Fischer) Let $V$ be a Hilbert space, and $e_1, e_2, \ldots$ its orthonormal basis. Then**
(1) **Fourier series.** For every \( v \in V \),
\[
v = (v,e_1)e_1 + (v,e_2)e_2 + \ldots
\]
i.e. the series is absolutely convergent and it converges to \( v \).

(2) **Parsevals’ identity.** For every \( v \in V \),
\[
||v||^2 = (v,e_1)^2 + (v,e_2)^2 + \ldots
\]
(3) If \((x_1, x_2, \ldots)\) is a sequence of real numbers such that
\[
x_1^2 + x_2^2 + \ldots < \infty
\]
then the series
\[
x_1e_1 + x_2e_2 + \ldots
\]
is absolutely convergent and it converges to an element in \( V \).

**Proof.**

(1) Let \( v_n = (v,e_1)e_1 + \ldots + (v,e_n)e_n \), for \( n \in \mathbb{N} \). We need to show that this sequence converges to \( v \). By the Bessel’s inequality, the series \( \sum_{n=0}^{\infty} (v,e_n)^2 \) is convergent. Thus, for every \( \epsilon > 0 \) there exists \( N \) such that
\[
\sum_{n>N} (v,e_n)^2 < \epsilon.
\]

If \( m > n > N \) then
\[
||v_m - v_n||^2 = (v,e_{n+1})^2 + \ldots + (v,e_m)^2 < \epsilon.
\]
This shows that the sequence \( (v_n) \) is Cauchy. Since \( V \) is complete, it has a limit \( \lim_{n} v_n = w \in V \). It remains to show that \( v = w \). Observe that, using the exercise,
\[
(v,e_i) = (\lim_{n} v_n,e_i) = \lim_{n} (v_n,e_i) = (v,e_i).
\]
Hence \( w - v \) is perpendicular to all \( e_i \) and to the linear span \( U \) of \( e_i \). But this space is dense, hence \( w - v = 0 \), as we proved in the last lecture.

(2) follows form
\[
||v||^2 = \lim_n ||v_n||^2,
\]
since \( v = \lim_n v_n \), and \( ||v_n||^2 = (v,e_1)^2 + \ldots + (v,e_n)^2 \).

(3) Let \( v_n = x_1e_1 + \ldots + x_n e_n \). Then \( (v_n) \) is Cauchy by the same argument as in (1) thus the series is converging to an element in \( V \) since \( V \) is complete.

\[\square\]

**Corollary 0.3.** Any Hilbert space \( V \) is isomorphic to \( \ell^2(\mathbb{N}) \).

**Proof.** Indeed, by the map
\[
v \mapsto ((v,e_1),(v,e_2),\ldots),
\]
for every \( v \in V \), is a norm-preserving isomorphism from \( V \) onto \( \ell^2(\mathbb{N}) \).

\[\square\]

This is a great result, for it gives a classification of Hilbert spaces. There is only one, up to an isomorphism.