MATH 5210, LECTURE 7 - HILBERT SPACES
APRIL 01

Let V be Euclidean space, that is, a vector space with a scalar product $(\cdot,\cdot) : V \times V \to \mathbb{R}$. Two vectors $x, y \in V$ are said to be orthogonal (perpendicular) if $(x, y) = 0$.

Exercise. (Pythagora) If x and y are orthogonal, then

$$||x + y||^2 = ||x||^2 + ||y||^2.$$

Solution:

$$||x+y||^2 = (x+y, x+y) = (x, x)+(x, y)+(y, x)+(y, y) = (x, x)+(y, y) = ||x||^2+||y||^2.$$

Henceforth we assume that V is infinite dimensional and separable, i.e. it contains a dense countable set S. We order S in any way:

$$s_1, s_2, \ldots$$

We perform the following sieve process to S: Cross s_n if it is linear combination of $s_1, s_2, \ldots, s_{n-1}$. In other words, we cross out s_1 if it is 0, s_2 if it is a multiple of s_1, etc. We arrive to a linearly independent sub-sequence

$$u_1, u_2, \ldots$$

of S. Let $U \subset V$ be the linear span of $u_1, u_2, \ldots$. This space contains S, hence it is dense in V. Thus U is a countably dimensional dense subspace of V. Conversely, if U is a dense, countably dimensional vector subspace of V with a basis $u_1, u_2, \ldots$ then the set of linear combinations

$$a_1u_1 + a_2u_1 + \ldots$$

where $a_1, a_2 \ldots \in \mathbb{Q}$ and almost all $a_i = 0$, is a countable dense subset S of V. Thus for Euclidean spaces, and more generally normed spaces, a more convenient way to define separability is via a dense, countably dimensional subspace. For example, if $V = L^2([0, 1])$ then the space of polynomial functions is a dense, countable dimensional subspace. We shall need the following lemma:

Lemma 0.1. *Let V be a Euclidean space and U a dense countably dimensional subspace. Let $v \in V$ such that $(v, u) = 0$ for all $u \in U$. Then $v = 0$.*

Proof. Since U is dense, there exists a convergent sequence (v_n) in U such that $\lim_n v_n = v$. By the Cauchy-Schwarz inequality,

$$|(v - v_n, v)| \le ||v - v_n|| \cdot ||v||.$$

Since $(v_n, v) = 0$, as v_n are in U thus perpendicular to v, the left hand side is the constant $||v||^2$. Since $\lim_n v_n = v$, $\lim_n ||v_n - v|| = 0$, and the right hand side goes to 0. Hence $||v|| = 0$, so $v = 0$. □

Let U be a dense countably dimensional subspace of V, and $u_1, u_2, \ldots$ a basis of U. We can perform the Gramm-Schmidt orthogonalization procedure to $u_1, u_2, \ldots$,

$$\begin{aligned} f_1 &= u_1 \\ f_2 &= u_2 - \tfrac{(u_2,f_1)}{(f_1,f_1)} f_1 \\ f_3 &= u_3 - \tfrac{(u_3,f_1)}{(f_1,f_1)} f_1 - \tfrac{(u_3,f_2)}{(f_2,f_2)} f_2 \\ &\vdots \end{aligned}$$

followed by normalization

$$e_i = \frac{f_i}{||f_i||}$$

to get an ortho-normal basis $e_1, e_2, \ldots$ of U, that its $(e_i, e_j) = 0$ if $i \neq j$ and 1 if $i = j$.

Our main goal is to write any $v \in V$ as a series

$$v = a_1 e_1 + a_2 e_2 + \ldots$$

where the right hand side is defined as the limit of the sequence of partial sums. Working formally, and multiplying both sides by e_i, we get that $a_i = (v, e_i)$ for all i. In the next lecture we shall prove that the series

$$(v, e_1)e_1 + (v, e_2)e_2 + \ldots$$

is absolutely converging and, assuming that V is complete, the series converges to v. Complete, separable Euclidean spaces are called Hilbert spaces. A set $e_1, e_2, \ldots$ of orthonormal vectors spanning a dense subset is called a basis of the Hilbert space.

An example of a Hilbert space is $L^2([a, b])$. Proof of completeness is similar to the one for $L^1([a, b])$, so we shall omit it. Furthermore, $L^2([a, b])$ is separable, since the subspace of polynomial functions is a dense countable dimensional subspace, just as it is in $L^1([a, b])$. In the special case $[a, b] = [-1, 1]$ the orthogonalization process applied to the basis $1, x, x^2, \ldots$ gives (multiples) of Legendre polynomials $P_n(x)$. Legendre polynomials are normalized so that $P_n(1) = 1$. Clearly $P_1(x) = 1$ and $P_2(x) = x$.

Exercise. Compute the third Legendre polynomial $P_3(x)$.

Solution. $P_3(x) = \frac{1}{2}(3x^2 - 1)$.