Let $V$ be Euclidean space, that is, a vector space with a scalar product $(\cdot, \cdot): V \times V \to \mathbb{R}$. Two vectors $x, y \in V$ are said to be orthogonal (perpendicular) if $(x, y) = 0$.

Exercise. (Pythagora) If $x$ and $y$ are orthogonal, then

$$||x + y||^2 = ||x||^2 + ||y||^2.$$ 

Henceforth we assume that $V$ is infinite dimensional and separable, i.e. it contains a dense countable set $S$. We order $S$ in any way:

$$s_1, s_2, \ldots$$

We perform the following sieve process to $S$: Cross $s_n$ if it is linear combination of $s_1, s_2, \ldots, s_{n-1}$. In other words, we cross out $s_1$ if it is 0, $s_2$ if it is a multiple of $s_1$, etc.

We arrive to a linearly independent sub-sequence

$$u_1, u_2, \ldots$$

of $S$. Let $U \subset V$ be the linear span of $u_1, u_2, \ldots$. This space contains $S$, hence it is dense in $V$. Thus $U$ is a countably dimensional dense subspace of $V$. Conversely, if $U$ is a dense, countably dimensional vector subspace of $V$ with a basis $u_1, u_2, \ldots$ then the set of linear combinations

$$a_1u_1 + a_2u_2 + \ldots$$

where $a_1, a_2, \ldots \in \mathbb{Q}$ and almost all $a_i = 0$, is a countable dense subset $S$ of $V$. Thus for Euclidean spaces, and more generally normed spaces, a more convenient way to define separability is via a dense, countably dimensional subspace. For example, if $V = L^2([0, 1])$ then the space of polynomial functions is a dense, countable dimensional subspace. We shall need the following lemma:

**Lemma 0.1.** Let $V$ be a Euclidean space and $U$ a dense countably dimensional subspace. Let $v \in V$ such that $(v, u) = 0$ for all $u \in U$. Then $v = 0$.

**Proof.** Since $U$ is dense, there exists a convergent sequence $(v_n)$ in $U$ such that $\lim_n v_n = v$. By the Cauchy-Schwarz inequality,

$$|(v - v_n, v)| \leq ||v - v_n|| \cdot ||v||.$$ 

Since $(v_n, v) = 0$, as $v_n$ are in $U$ thus perpendicular to $v$, the left hand side is the constant $||v||^2$. Since $\lim_n v_n = v$, $\lim_n ||v_n - v|| = 0$, and the right hand side goes to 0. Hence $||v|| = 0$, so $v = 0$. 

$\square$
Let $U$ be a dense countably dimensional subspace of $V$, and $u_1, u_2, \ldots$ a basis of $U$. We can perform the Gramm-Schmidt orthogonalization procedure to $u_1, u_2, \ldots$

\[
\begin{align*}
  f_1 &= u_1 \\
  f_2 &= u_2 - \left(\frac{(u_2, f_1)}{(f_1, f_1)}\right) f_1 \\
  f_3 &= u_3 - \left(\frac{(u_3, f_1)}{(f_1, f_1)}\right) f_1 - \left(\frac{(u_3, f_2)}{(f_2, f_2)}\right) f_2 \\
  &\vdots
\end{align*}
\]

followed by normalization

\[e_i = \frac{f_i}{\|f_i\|}\]

to get an ortho-normal basis $e_1, e_2, \ldots$ of $U$, that its $(e_i, e_j) = 0$ if $i \neq j$ and $1$ if $i = j$.

Our main goal is to write any $v \in V$ as a series

\[v = a_1 e_1 + a_2 e_2 + \ldots\]

where the right hand side is defined as the limit of the sequence of partial sums. Working formally, and multiplying both sides by $e_i$, we get that $a_i = (v, e_i)$ for all $i$. In the next lecture we shall prove that the series

\[(v, e_1)e_1 + (v, e_2)e_2 + \ldots\]

is absolutely converging and, assuming that $V$ is complete, the series converges to $v$. Complete, separable Euclidean spaces are called Hilbert spaces. A set $e_1, e_2, \ldots$ of orthonormal vectors spanning a dense subset is called a basis of the Hilbert space.

An example of a Hilbert space is $L^2([a, b])$. Proof of completeness is similar to the one for $L^1([a, b])$, so we shall omit it. Furthermore, $L^2([a, b])$ is separable, since the subspace of polynomial functions is a dense countable dimensional subspace, just as it is in $L^1([a, b])$. In the special case $[a, b] = [-1, 1]$ the orthogonalization process applied to the basis $1, x, x^2, \ldots$ gives (multiples) of Legendre polynomials $P_n(x)$. Legendre polynomials are normalized so that $P_n(1) = 1$. Clearly $P_1(x) = 1$ and $P_2(x) = x$.

Exercise. Compute the third Legendre polynomial $P_3(x)$. 