MATH 6370, LECTURE 6, ARTIN L-FUNCTIONS MARCH 30

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Let F be a Galois extension of $\mathbb{Q}$ of degree n and G the Galois group of F. Let A be the ring of integers in F. Let p be a prime. We have a factorization $Ap = P_1^e \cdot \ldots \cdot P_g^e$, let P be any of these primes. Then A/P is a degree f extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where $n = efg$. Recall that the decomposition group $D_P \subset G$ consists of all $\sigma \in G$ such that $\sigma(P) = P$. In particular D_P acts naturally on A/P. In fact we have proved that the natural action gives an exact sequence

$$1 \to I_P \to D_P \to \mathrm{Gal}(A/P) \to 1.$$

The Galois group $\mathrm{Gal}(A/P)$ is a cyclic group of order f, generated by the Frobenius element, raising to the p-th power.

Let (ρ, V) be a finite-dimensional representation of G, that is, a homomorphism

$$\rho : G \to GL(V)$$

where V is a finite-dimensional vector space over $\mathbb{C}$. In particular, if $V = \mathbb{C}$ then $GL(V) = \mathbb{C}^\times$, thus one-dimensional representations are characters. Every group G has a canonical representation, so called regular representation. More precisely, we set $V = \mathbb{C}[G]$, the set of complex valued functions on G. For every $f(x) \in \mathbb{C}[G]$ and $g \in G$ we define

$$(r(g)(f))(x) = f(xg),$$

in words, G acts by right translations on the space of functions on G.

Exercise. Prove that $r(gh) = r(g)r(h)$, i.e. r is a representation of G.

To every representation (ρ, V) we can attach an L-function (Artin). It is defined formally as a product over all primes p

$$L(\rho, s) = \prod_p L_p(\rho, s)$$

where $L_p(\rho, s)$ is defined as follows. Pick a prime ideal $P \subset A$ appearing in factorization of p. Let

$$V^{I_P} = \{v \in V : \rho(g)v = v \text{ for all } g \in I_P\}.$$

The group D_P/I_P naturally acts on V^{I_P}, let

$$\rho_P : D_P/I_P \to GL(V^{I_P})$$

be the resulting representation. Let $\mathrm{Fr}_P \in D_P/I_P \cong \mathrm{Gal}(A/P)$ be the Frobenius element. Then

$$L_p(\rho, s) = \frac{1}{\det\left(1 - \frac{1}{p^s}\rho_p(\mathrm{Fr}_P)\right)}.$$

Observe that this definition is independent of the choice of P dividing p, since all such P are conjugated by G, and determinant is conjugation invariant function. Also, if $V^{I_P} = 0$ then $L_p(\rho, s) = 1$.

Examples. Let ρ be the trivial representation, i.e. a trivial character. Then $V = V^{I_P} = \mathbb{C}$ for all P and $\rho_p(\mathrm{Fr}_P) = 1$. Hence

$$L_p(\rho, s) = \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

and the L-function is simply the Riemann zeta function. This is not that surprising since the trivial representation "forgets" the group G, and thus the field F.

As the next example, consider $F = \mathbb{Q}(\omega)$ where ω is the ℓ-th root of 1. Then $G \cong (\mathbb{Z}/\ell\mathbb{Z})^\times$ where the isomorphism sends $n \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ to $\sigma_n \in G$ defined by $\sigma_n(\omega) = \omega^n$. Since G is abelian I_P and D_P depend only on p and we adopt notation I_p and D_p. The only ramified prime is ℓ and, recall, $(\ell) = (1-\omega)^{\ell-1}$, so $e = \ell - 1$ in this case. Hence $I_\ell = G$ and $I_p = \{1\}$ otherwise. If $p \neq \ell$ then $\mathrm{Fr}_p = \sigma_p$. Thus, if χ is a non-trivial character of G then $L_\ell(\chi, s) = 1$ while for $p \neq \ell$,

$$L_p(\rho, s) = \frac{1}{\left(1 - \frac{\chi(p)}{p^s}\right)}$$

It follows that the Artin L-function is the Dirichlet's L-function in this case.

A representation (ρ, V) is called reducible if $V = V_1 \oplus V_2$ and for any $g \in G$ the operator $\rho(g)$ preserves V_1 and V_2. Let $\rho_1(g)$ and $\rho_2(g)$ be the linear transformations obtained by restricting $\rho(g)$ on V_1 and V_2, respectively. Then $\rho(g)$ is a block diagonal matrix with diagonal blocks $\rho_1(g)$. Since the determinant of a block diagonal matrix is a product of determinants of the diagonal blocks, it is evident that

$$L(\rho, s) = L(\rho_1, s)L(\rho_2, s).$$

Example: Assume that G is abelian. Then the characters χ of G form a basis of $\mathbb{C}[G]$. Hence

$$\mathbb{C}[G] = \oplus_\chi \mathbb{C} \cdot \chi.$$

Every line $\mathbb{C} \cdot \chi$ is preserved by the action of G, since for every $g \in G$,

$$(r(g)(\chi))(x) = \chi(xg) = \chi(g) \cdot \chi(x)$$

for all $x \in G$.

In the next lecture we shall show that the Artin L-function attached to the regular representation of G is the Dedekind zeta function of F. As a consequence, in the special case of cyclotomic field $F = \mathbb{Q}(\omega)$, we get that the Dedekind zeta function is the product of the Dirichlet L-functions attached to characters mod ℓ, a key result used in the proof of the theorem on primes in progression.