MATH 5210, LECTURE 6 - EUCLIDEAN SPACES
MARCH 30

Euclidean space is a vector space V over $\mathbb{R}$ with a scalar product. Scalar product is a function $(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that

- $(x, y) = (y, x)$
- $(\lambda x, y) = \lambda(x, y) = (x, \lambda y)$
- $(x, y + z) = (x, y) + (x, z)$.
- $(x, x) \geq 0$ and it is 0 if and only if $x = 0$.

for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$. The classical example is the dot product on $V = \mathbb{R}^k$, the space of all k-tuples of real numbers:

$$(x, y) = \sum_{i=1}^{k} x_i y_i$$

where $x = (x_i)$ and $y = (y_i)$ are two elements in V. Any Euclidean space is a normed space for the norm

$$||x|| = \sqrt{(x, x)}.$$

Of course, we need to verify that $||x||$ satisfies three norm axioms. The only nontrivial part is to check the triangle inequality. To that end we need the Cauchy-Schwarz inequality:

Lemma 0.1. *For any $x, y \in V$,*

$$|(x, y)| \leq ||x|| \cdot ||y||.$$

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(\lambda) = (x + \lambda y, x + \lambda y) \geq 0,$$

where ≥ 0 is a consequence of the fourth bullet. Using the first three bullets, we can rewrite

$$f(\lambda) = (x, x) + 2\lambda(x, y) + \lambda^2(y, y) = c + b\lambda + a\lambda^2.$$

Thus f is a quadratic polynomial whose graph is a parabola in the upper half plane since $f \geq 0$. In particular, f does not have two different real roots. This implies that the discriminant of the polynomial is not positive:

$$b^2 - 4ac = 4(x, y)^2 - 4(x, x)(y, y) \leq 0.$$

Lemma follows after taking square roots. □

Now it is easy to prove the triangle inequality:

$$||x+y||^2 = (x+y, x+y) = (x,x) + 2(x,y) + (y,y) \leq (||x|| + ||y||)^2$$

where we substituted $(x,x) = ||x||^2$, $(y,y) = ||y||^2$ and used the Cauchy-Schwarz inequality for the last step.

Examples:

1) Let $\ell^2(\mathbb{N})$ be the set of infinite tuples $x = (x_1, x_2, \ldots)$ of real numbers such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty.$$

If $x = (x_1, x_2, \ldots)$ is in $\ell^2(\mathbb{N})$, then $\lambda x = (\lambda x_1, \lambda x_2, \ldots)$ is also in $\ell^2(\mathbb{N})$.

Exercise. If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are in $\ell^2(\mathbb{N})$, show that $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$ is in $\ell^2(\mathbb{N})$.

Solution. We need to show that $\sum_{i=1}^{\infty}(x_i + y_i)^2$ is convergent. By the Cauchy-Schwarz inequality, $(x_i + y_i)^2 = x_i^2 + 2x_iy_i + y_i^2 \leq 2(x_i^2 + y_i^2)$ hence

$$\sum_{i=1}^{\infty}(x_i + y_i)^2 \leq 2\sum_{i=1}^{\infty} x_i^2 + 2\sum_{i=1}^{\infty} y_i^2 < \infty.$$

It follows that $\ell^2(\mathbb{N})$ is a vector space. The scalar product is defined by

$$(x,y) = \sum_{i=1}^{\infty} x_iy_i$$

This series is absolutely convergent since, for every n,

$$\sum_{i=1}^{n} |x_iy_i| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}$$

by the Cauchy-Schwarz inequality for $\mathbb{R}^n$.

2) $C([0,1])$, the space of continuous functions on $[0,1]$. The scalar product is

$$(f,g) = \int_0^1 f(t)g(t)\ dt.$$

2) $L^2([0,1])$, the space of equivalence classes of Lebesgue measurable functions f on $[0,1]$ such that f^2 is integrable. Assume f and g are two such functions. Expanding $(f \pm g)^2 \geq 0$ gives

$$|fg| \leq \frac{f^2 + g^2}{2}.$$

It follows that fg is Lebesgue integrable, since it is measurable and bounded by an integrable function. Hence on $L^2([0,1])$ we have a well defined scalar product

$$(f,g) = \int fg.$$

This space is isomorphic to the completion of example 2).