Let $V$ be a vector space over $\mathbb{R}$. Scalar product on $V$ is a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ such that

- $(x, y) = (y, x)$
- $(\lambda x, y) = \lambda (x, y) = (x, \lambda y)$
- $(x, y + z) = (x, y) + (x, z)$.
- $(x, x) \geq 0$ and it is 0 if and only if $x = 0$.

for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$. The classical example is the dot product on $V = \mathbb{R}^k$, the space of all $k$-tuples of real numbers:

$$(x, y) = \sum_{i=1}^{k} x_i y_i$$

where $x = (x_i)$ and $y = (y_i)$ are two elements in $V$. Any Euclidean space is a normed space for the norm

$$||x|| = \sqrt{(x, x)}.$$

Of course, we need to verify that $||x||$ satisfies three norm axioms. The only nontrivial part is to check the triangle inequality. To that end we need the Cauchy-Schwarz inequality:

Lemma 0.1. For any $x, y \in V$,

$$|(x, y)| \leq ||x|| \cdot ||y||.$$

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(\lambda) = (x + \lambda y, x + \lambda y) \geq 0,$$

where $\geq 0$ is a consequence of the fourth bullet. Using the first three bullets, we can rewrite

$$f(\lambda) = (x, x) + 2\lambda (x, y) + \lambda^2 (y, y) = c + b\lambda + a\lambda^2.$$

Thus $f$ is a quadratic polynomial whose graph is a parabola in the upper half plane since $f \geq 0$. In particular, $f$ does not have two different real roots. This implies that the discriminant of the polynomial is not positive:

$$b^2 - 4ac = 4(x, y)^2 - 4(x, x)(y, y) \leq 0.$$

Lemma follows after taking square roots. \qed
Now it is easy to prove the triangle inequality:

\[ ||x + y||^2 = (x + y, x + y) = (x, x) + 2(x, y) + (y, y) \leq (||x|| + ||y||)^2 \]

where we substituted \((x, x) = ||x||^2, (y, y) = ||y||^2\) and used the Cauchy-Schwarz inequality for the last step.

Examples:
1) Let \(\ell^2(\mathbb{N})\) be the set of infinite tuples \(x = (x_1, x_2, \ldots)\) of real numbers such that

\[ \sum_{i=1}^{\infty} x_i^2 < \infty. \]

If \(x = (x_1, x_2, \ldots)\) is in \(\ell^2(\mathbb{N})\), then \(\lambda x = (\lambda x_1, \lambda x_2, \ldots)\) is also in \(\ell^2(\mathbb{N})\).

Exercise. If \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) are in \(\ell^2(\mathbb{N})\), show that \(x + y = (x_1 + y_1, x_2 + y_2, \ldots)\) is in \(\ell^2(\mathbb{N})\).

It follows that \(\ell^2(\mathbb{N})\) is a vector space. The scalar product is defined by

\[ (x, y) = \sum_{i=1}^{\infty} x_i y_i \]

This series is absolutely convergent since, for every \(n\),

\[ \sum_{i=1}^{n} |x_i y_i| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2} \]

by the Cauchy-Schwarz inequality for \(\mathbb{R}^n\).

2) \(C([0, 1])\), the space of continuous functions on \([0, 1]\). The scalar product is

\[ (f, g) = \int_{0}^{1} f(t)g(t) \, dt. \]

2) \(L^2([0, 1])\), the space of equivalence classes of Lebesgue measurable functions \(f\) on \([0, 1]\) such that \(f^2\) is integrable. Assume \(f\) and \(g\) are two such functions. We have the following AM-GM inequality:

\[ |fg| \leq \frac{f^2 + g^2}{2}. \]

It follows that \(fg\) is Lebesgue integrable, since it is measurable and bounded by an integrable function. Hence on \(L^2([0, 1])\) we have a well defined scalar product

\[ (f, g) = \int fg. \]

This space is isomorphic to the completion of example 2).