Let \( \ell \) be a prime, and \( a \) an integer prime to \( \ell \). Let \( S \) be the set of primes \( p \equiv a \pmod{\ell} \). We shall apply results from the previous two lectures to prove that \( S \) has Dirichlet density \( 1/(\ell - 1) \). Recall that the Dirichlet density is the limit

\[
\delta(S) := \lim_{s \to 1^+} \frac{\sum_{p \in S} \frac{1}{p^s}}{\sum_{p \in X} \frac{1}{p^s}},
\]

if it exists, where \( X \) is the set of all primes. Consider \( a \) as an element of the abelian group \( G = (\mathbb{Z}/\ell\mathbb{Z})^\times \). Let \( \delta_a : G \to \mathbb{C} \) be the characteristic function of \( a \), that is,

\[
\delta_a(b) = \begin{cases} 
1 & \text{if } a = b \\
0 & \text{otherwise}.
\end{cases}
\]

We can view \( \delta_a \) as a periodic function on \( \mathbb{Z} \), by defining \( \delta_a(n) = 0 \) for all \( n \) divisible by 0. In this way the above formula can be rewritten as

\[
\delta(S) = \lim_{s \to 1^+} \frac{\sum_{p \in X} \delta_a(p)}{\sum_{p \in X} \frac{1}{p^s}}.
\]

We shall now apply the Fourier transform to express \( \delta_a \) as a linear combination of characters of \( G \). Recall that

\[
\delta_a = \sum_{\chi \in \hat{G}} \hat{\delta}_a(\chi) \cdot \chi
\]

where

\[
\hat{\delta}_a(\chi) = \frac{1}{|G|} \sum_{x \in G} \delta_a(x) \overline{\chi(x)} = \frac{1}{\ell - 1} \overline{\chi(a)}.
\]

Thus

\[
\delta_a = \frac{1}{\ell - 1} \sum_{\chi \in \hat{G}} \chi(a) \cdot \chi
\]

and we can write

\[
\sum_{p \in X} \frac{\delta_a(p)}{p^s} = \frac{1}{\ell - 1} \sum_{\chi \in \hat{G}} \chi(a) \sum_{p \in X} \frac{\chi(p)}{p^s}.
\]

If \( \chi \neq 1 \) then we proved that

\[
\lim_{s \to 1^+} \frac{\sum_{p \in X} \frac{\chi(p)}{p^s}}{\sum_{p \in X} \frac{1}{p^s}} = 0,
\]

hence \( \delta(S) = 1/(\ell - 1) \).
We now discuss relationship with splitting of primes in cyclotomic extensions. Working generally, let $F$ be a Galois extension of $\mathbb{Q}$ of degree $n$. Let $A$ be the ring of integers. Let $p$ be an unramified prime. Then we have a factorization $Ap = P_1 \cdot \ldots \cdot P_g$, where these primes are mutually different. Let $P$ one of these primes. Then $A/P$ is a degree $f$ extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where $n = fg$. Recall that the decomposition group $D_P \subset G$ consists of all $\sigma \in G$ such that $\sigma(P) = P$. In particular $D_P$ acts naturally on $A/P$. In fact we have proved that the natural action gives an isomorphism

$$D_P \cong \text{Gal}(A/P).$$

The Galois group $\text{Gal}(A/P)$ is a cyclic group of order $f$, generated by the Frobenius element, raising to the $p$-th power. In view of the isomorphism there exists a unique element $\text{Fr}_P \in D_P$ such that

$$\text{Fr}_P(x) \equiv x^p \pmod{P}$$

for all $x \in A$. Since $G$ acts transitively on the primes $P_1, \ldots, P_g$,

$$\text{Fr}_p = \{\text{Fr}_{P_1}, \ldots, \text{Fr}_{P_g}\}$$

is a conjugacy class in $G$, the Frobenius class of $p$. If $G$ is abelian, every conjugacy class is a singleton, hence we have a proper (Frobenius) element $\text{Fr}_p$ in $G$.

Let’s work this out for the case $F = \mathbb{Q}(\omega)$ where $\omega$ is $\ell$-th root of 1. Then $G \cong (\mathbb{Z}/\ell\mathbb{Z})^\times$ where $a \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ gives $\sigma_a \in G$ defined by $\sigma_a(\omega) = \omega^a$. All primes $p \neq \ell$ are unramified, and it is clear that $\text{Fr}_p = \sigma_p$. Thus, if we fix $\sigma$, which is the same as fixing $a$, then the set $S$ of primes $p \neq \ell$ such that $\text{Fr}_p = \sigma$ is the same as the set of primes $p \equiv a \pmod{\ell}$. In particular, $S$ has Dirichlet density $1/(\ell - 1)$. This is a special case of the Čebotarev density theorem:

**Theorem 0.1.** Let $F$ be a Galois extension of $\mathbb{Q}$ with (finite) Galos group $G$. Fix a conjugacy class $C \subset G$. Let $S$ be the set of unramified primes such that the Frobenius class is $C$. Then

$$\delta(S) = \frac{|C|}{|G|}.$$