MATH 5210, LECTURE 5 - HAHN-BANACH THEOREM
MARCH 27

In this lecture we shall prove the following fundamental result (the Hahn-Banach Theorem) about normed spaces:

Theorem 0.1. *Let V_0 be a subspace of a normed space V. Let $f_0 : V_0 \to \mathbb{R}$ be a bounded linear functional i.e. there exists $C \geq 0$ such that*

$$|f_0(x)| \leq C||x||$$

for all $x \in V_0$. Then f_0 can be extended to a linear functional $f : V \to \mathbb{R}$ satisfying the same bound.

We shall prove this theorem assuming that V is separable i.e. it contains a dense countable subset. This assumption lets us avoid use of Zorn's lemma. Most "naturally" occurring normed spaces are separable. For example, consider the space $C([0,1])$ of continuous functions on $[0,1]$ with the sup norm. By the theorem of Stone-Weierstrass, the space of polynomial functions is dense. Moreover:

Exercise. Let $p(x) = a_n x^n + \ldots + a_0$ be a polynomial. Show that, for every $\epsilon > 0$, there exists a polynomial $q(x) = b_n x^n + \ldots + b_0$ with rational coefficients b_i such that

$$\sup_{x \in [0,1]} |p(x) - q(x)| < \epsilon.$$

Solution. Pick rational numbers such that $|a_i - b_i| < \epsilon/(n+1)$, for all $i = 0, \ldots, n$. Then, for every $x \in [0,1]$ we have $|x|^i \leq 1$, for all i. Hence

$$|p(x) - q(x)| \leq |a_n = b_n|x|^n + \ldots + |a_0 - b_0| < (n+1) \cdot \frac{\epsilon}{n+1} = \epsilon.$$

(This also implies that the supremum is less than ϵ, why?).

Thus the set of polynomials with rational coefficients (a countable set) is dense in $C([0,1])$, hence $C([0,1])$ is separable. The space $L^1([0,1])$ is a also separable. Indeed, we have shown that $C([0,1])$ is dense in $L^1([0,1])$ and thus the set of polynomials with rational coefficients is dense in $L^1([0,1])$.

We go on to prove the theorem. Observe that it suffices to construct a functional f such that $f(x) \leq C||x||$ for all $x \in V$. Indeed, the we also have $f(-x) \leq C||-x||$. But $f(-x) = -f(x)$, since f is linear, and $||-x|| = ||x||$. Hence both

$$f(x) \leq C||x|| \text{ and } -f(x) \leq C||x||$$

hold, and this is equivalent to $|f(x)| \leq C||x||$. Let $z \in V$. We shall first extend f_0 to $V_0 + \mathbb{R}z$. To that end, we need the following: For every $x, y \in V_0$, we have

$$f_0(x) + f_0(y) = f_0(x+y) \leq C||x+y|| \leq C||x-z|| + C||y+z||,$$

where the second is the triangular inequality. Rearranging, we get

$$f_0(x) - C||x-z|| \leq -f_0(y) + C||y+z||$$

for all $x, y \in V_0$. It follows that the supremum, over all $x \in V_0$, of the numbers on the left side, is less than or equal to the infimum, over all $y \in V_0$, of the numbers on the right side. Hence there exists a real number γ such that

$$f_0(x) - C||x-z|| \leq \gamma \leq -f_0(y) + C||y+z||.$$

for all $x, y \in V_0$.

Now we can extend f to $V_0 + \mathbb{R}z$. If $z \in V_0$, then there is nothing to prove. Otherwise any element in $V_0 + \mathbb{R}z$ can be uniquely written as $v + tz$ for $v \in V_0$ and $t \in \mathbb{R}$. Let

$$f(v+tz) = f_0(v) + t\gamma.$$

We need to check that

$$f_0(v) + t\gamma \leq C||v+tz||.$$

If $t = 0$, this holds by the assumption on f_0. If $t > 0$, we shall divide this inequality by t, if $t < 0$ we shall divide by $u = -t$. If $t > 0$ then

$$f_0(v/t) + \gamma \leq C||v/t + z||$$

or

$$\gamma \leq -f_0(v/t) + C||v/t + z||.$$

If $t < 0$, then

$$f_0(v/u) - \gamma \leq C||v/u - z||$$

or

$$f_0(v/u) - C||v/u - z|| \leq \gamma.$$

These inequalities hold by the choice of γ. Thus we have extended f_0 to $V_0 + \mathbb{R}z$.

Since we assume that V is separable, there exists a dense countable set $S = \{z_1, z_2, \ldots\}$ in V. We define a sequence of subspaces

$$V_1 = V_0 + \mathbb{R}z_1 \subseteq V_2 = V_1 + \mathbb{R}z_2 \subseteq \ldots.$$

Following the above procedure, one step at a time, we can extend f_0 to the union $U = \cup_{n=1}^{\infty} V_n$ of these spaces. Let

$$g : U \to \mathbb{R}$$

denote this extension. By construction, we have $|g(x)| \leq C||x||$ for all $x \in U$. We also know that U is dense in V. By HW 3 exercise 4), g extends to a functional f on V satisfying the same bound.

An important consequence of the Hahn Banach Theorem is that continuous (i.e. bounded) functionals on a normed space V separate points:

Corollary 0.2. *Let V be a normed vectors space and $x \neq y$ two elements in V. Then there exists a continuous functional f on V such that $f(x) \neq f(y)$.*

Proof. Let $z = x - y$ and $V_0 = \mathbb{R}z$. Let $f_0 : V_0 \to \mathbb{R}$ be the linear functional defined by $f_0(z) = 1$. Then $f_0(x) \neq f_0(y)$ and f extends to V. □