This lecture could be titled Fourier analysis on finite abelian groups.

A character of $G$ is a homomorphism $\chi : G \rightarrow \mathbb{C}^\times$. A product of any two characters is a character, and $\chi^{-1}(g) := \chi(g)^{-1}$ is a character. Thus the set of all characters form a commutative group denoted by $\hat{G}$. This group can be trivial, for example, if $G$ is simple. Assume now that $G$ is finite and commutative. We shall prove that $\hat{G}$ is isomorphic to $G$ in this case, however, this isomorphism is not canonical. Assume first that $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group. Then $\chi$ is determined by $\chi(1)$ and this number can be any $n$-th root of 1. Let $\mu_n \subset \mathbb{C}^\times$ be the group of complex $n$-th roots of 1 i.e. the set of all complex numbers $z$ such that $z^n = 1$. Thus $\hat{G} \cong \mu_n$ by the map $\chi \mapsto \chi(1)$. The group $\mu_n$ is also a cyclic group of order $n$, thus it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, but this isomorphism depends on the choice of the generator of $\mu_n$, i.e., a primitive $n$-th root of 1. Summarizing, we proved $G \cong \hat{G}$ for cyclic groups. The general case follows from classification of finite abelian groups: There exists integers $d_1 | d_2 | \cdots | d_k$ such that

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}.$$ 

Exercise: Let $G_1$ and $G_2$ be two finite abelian groups. Prove that

$$G_1 \hat{\times} G_2 \cong \hat{G}_1 \times \hat{G}_2.$$ 

It follows that

$$\hat{G} \cong \mu_{d_1} \times \mu_{d_2} \times \cdots \times \mu_{d_k} \cong G.$$ 

Let $\mathbb{C}[G]$ be the vector space of complex valued functions on $G$. Clearly the dimension of this space is the order of the group $G$. We know, from a homework exercise, that characters are linearly independent functions on $G$. Since the number of characters is equal to the order of $|G|$, it follows that the characters give a basis of $\mathbb{C}[G]$. Thus every $f \in \mathbb{C}[G]$ can be written as

$$f(g) = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \cdot \chi(g)$$

for some $\hat{f}(\chi) \in \mathbb{C}$.

Exercise: Prove that the characters are orthonormal for the dot product on $\mathbb{C}[G]$ defined by

$$(\psi, \varphi) = \frac{1}{|G|} \sum_{g \in G} \psi(g) \bar{\varphi}(g)$$

where $\psi, \varphi \in \mathbb{C}[G]$ and $\bar{\varphi}$ denotes the complex conjugate.
Using orthonormality of characters, we can now determine the coefficients $\hat{f}(\chi)$ using the dot product:

$$\hat{f}(\chi) = (f, \chi) = \frac{1}{|G|} \sum_{g \in G} f(g) \bar{\chi}(g).$$

This formula explains the notation, $\hat{f}(\chi)$ is a function on $\hat{G}$, the Fourier transform of $f$.

The group $\hat{G}$ is called the dual group. As a side remark, we mention that the dual group of $\hat{G}$ is isomorphic to $G$. Indeed, $g \in G$ defines as a character of $\hat{G}$ by $\chi \mapsto \chi(g)$ for all $\chi \in \hat{G}$. This is called Pontrjagin duality.

Exercise: Prove that the natural map $G \to \hat{\hat{G}}$ defined above is an isomorphism. Hint: it suffices to show that it is injective, why?