MATH 6370, LECTURE 4
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GORDAN SAVIN

This lecture could be titled Fourier analysis on finite abelian groups.

A character of G is a homomorphism $\chi : G \to \mathbb{C}^\times$. A product of any two characters is a character, and $\chi^{-1}(g) := \chi(g)^{-1}$ is a character. Thus the set of all characters form a commutative group denoted by $\hat{G}$. This group can be trivial, for example, if G is simple. Assume now that G is finite and commutative. We shall prove that $\hat{G}$ is isomorphic to G in this case, however, this isomorphism is not canonical. Assume first that $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group. Then χ is determined by $\chi(1)$ and this number can be any n-th root of 1. Let $\mu_n \subset \mathbb{C}^\times$ be the group of complex n-th roots of 1 i.e. the set of all complex numbers z such that $z^n = 1$. Thus $\hat{G} \cong \mu_n$ by the map $\chi \mapsto \chi(1)$. The group μ_n is also a cyclic group of order n, thus it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, but this isomorphism depends on the choice of the generator of μ_n i.e. a primitive n-th root of 1. Summarizing, we proved $G \cong \hat{G}$ for cyclic groups. The general case follows form classification of finite abelian groups: There exists integers $d_1 \mid d_2 \mid \cdots \mid d_k$ such that

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}.$$

Exercise: Let G_1 and G_2 be two finite abelian groups. Prove that

$$\widehat{G_1 \times G_2} \cong \hat{G}_1 \times \hat{G}_2.$$

It follows that

$$\hat{G} \cong \mu_{d_1} \times \mu_{d_2} \times \cdots \times \mu_{d_k} \cong G.$$

Let $\mathbb{C}[G]$ be the vector space of complex valued functions on G. Clearly the dimension of this space is the order of the group G. We know, from a homework exercise, that characters are linearly independent functions on G. Since the number of characters is equal to the order of $|G|$, it follows that the characters give a basis of $\mathbb{C}[G]$. Thus every $f \in \mathbb{C}[G]$ can be written as

$$f(g) = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \cdot \chi(g)$$

for some $\hat{f}(\chi) \in \mathbb{C}$.

Exercise: Prove that the characters are orthonormal for the dot product on $\mathbb{C}[G]$ defined by

$$(\psi, \varphi) = \frac{1}{|G|} \sum_{g \in G} \psi(g)\bar{\varphi}(g)$$

where $\psi, \varphi \in \mathbb{C}[G]$ and $\bar{\varphi}$ denotes the complex conjugate.

Using orthonormality of characters, we can now determine the coefficients $\hat{f}(\chi)$ using the dot product:

$$\hat{f}(\chi) = (f, \chi) = \frac{1}{|G|} \sum_{g \in G} f(g) \bar{\chi}(g).$$

This formula explains the notation, $\hat{f}(\chi)$ is a function on $\hat{G}$, the Fourier transform of f.

The group $\hat{G}$ is called the dual group. As a side remark, we mention that the dual group of $\hat{G}$ is isomorphic to G. Indeed, $g \in G$ defines as a character of $\hat{G}$ by $\chi \mapsto \chi(g)$ for all $\chi \in \hat{G}$. This is called Pontrjagin duality.

Exercise: Prove that the natural map $G \to \hat{\hat{G}}$ defined above is an isomorphism. Hint: it suffices to show that it is injective, why?