MATH 5210, LECTURE 4 - CONTINUOUS LINEAR TRASFORMATIONS MARCH 25

In linear algebra we study vector spaces and maps between them $T : V \to U$ that preserve vector space structure (linear transformations)

$$T(x+y) = T(x) + T(y) \text{ and } T(\lambda x) = \lambda T(x)$$

for all $x, y \in V$ and $\lambda \in \mathbb{R}$. On the other hand, normed spaces come with norms that turn vector spaces into metric spaces, so it is natural to look at continuous linear transformations. To that end we make the following definition: T is bounded if there exists $C \geq 0$ such that

$$||T(x)|| \leq C||x||$$

for every $x \in V$. This inequality implies that T maps the ball of radius r centered at 0 into the ball of radius rC centered at 0. In particular, T maps bounded sets to bounded sets. Assume that $x \neq 0$. Then, after dividing by $||x||$, the above can be rewritten as

$$\frac{1}{||x||}||T(x)|| = ||T(\frac{x}{||x||})|| \leq C$$

where we used linearity of T to derive the left equality. Observe that $x/||x||$ is element of norm 1, so T bounded is equivalent to demanding that

$$\sup_{||x||=1} ||T(x)|| = \sup_{||x||\leq 1} ||T(x)|| = \sup_{||x||<1} ||T(x)||$$

is finite.

Exercise: Explain the above equalities. Hint: $||x|| = \lim_{\lambda\to 1^-} ||\lambda x||$.

Solution: Observe that, for $0 < \lambda < 1$ and $||x|| = 1$ we have $||T(\lambda x)|| = \lambda \cdot ||T(x)||$ hence

$$\sup_{0<\lambda<1} ||T(\lambda x)|| = ||T(x)||.$$

This implies

$$\sup_{||x||<1} ||T(x)|| = \sup_{||x||\leq 1} ||T(x)||.$$

Let's look at some examples. Linear transformations of finite dimensional vector spaces are bounded. To see this, let $V = \mathbb{R}^n$ with the norm $||x|| = \sup(|x_1|, \ldots, |x_n|)$, for every $x = (x_1, \ldots, x_n) \in V$. A map $T : V \to V$ is given by an $n \times n$ matrix

$A = (a_{ij})$. Let c be the maximum of $|a_{ij}|$. Let $x \in V$ be such that $||x|| \leq 1$. Let $y = T(x)$, $y = (y_1, \ldots, y_n)$. Then

$$|y_i| = |a_{i1}x_1 + \ldots + a_{in}x_n| \leq |a_{i1}x_1| + \ldots + |a_{in}x_n| \leq nc.$$

Hence T is bounded with $C = nc$. On the other hand, it is easy to construct unbounded linear operators on infinite dimensional spaces. Let V be the space of infinite tuples $x = (x_1, x_2, \ldots)$ such that $x_n = 0$ for all but finitely many entries. This is also a normed space with norm $||x|| = \sup_n |x_n|$. Then $T : V \to V$ defined by

$$T((x_1, x_2, x_3, \ldots)) = (1x_1, 2x_2, 3x_3, \ldots).$$

is unbounded, check it.

Proposition 0.1. *Let $T : V \to U$ be a linear transformation of normed linear vector spaces. Then T is bounded if and only if it is continous.*

Proof. Assume T is bounded, i.e. there exists $C \geq 0$ such that $||T(x)|| \leq C||x||$ for all $x \in V$. Then

$$||T(x) - T(y)|| = ||T(x - y)|| \leq C||x - y||.$$

Let $\epsilon > 0$ and set $\delta = \epsilon/C$. If $||x - y|| < \delta$ then $||T(x) - T(y)|| < \epsilon$ by the inequality. Hence T is (uniformly) continuous. Conversely, assume that T is continuous. In fact, it will enough to assume that T is continuous at 0. Recall that $T(0) = 0$. Then, given $\epsilon > 0$ there exists $\delta > 0$ such that $||T(x)|| < \epsilon$ if $||x|| < \delta$. Divide both by δ and substitute $y = x/\delta$. Then

$$||T(y)|| < \frac{\epsilon}{\delta}$$

for all $||y|| < 1$. Hence T is bounded. □

Of special interest are linear maps $\ell : V \to \mathbb{R}$. These are called linear functionals. In this case bounded (continuous) means there exists $C \geq 0$ such that

$$|\ell(v)| \leq C||v||$$

for all $v \in V$.

Examples: Let $V = C([0, 1])$ be the space of continuous function, with the sup norm: $||f|| = \sup_{x\in[0,1]} |f(x)|$. Then

$$\ell(f) = f(0),$$

evaluation functions at 0, is a bounded linear functional. What is C? (Instead of 0 we can take any other element in $[0, 1]$.) Next, consider the functional

$$\ell(f) = \int_0^1 f(t)\ dt.$$

Since $|f(t)| \leq ||f||$,

$$|\ell(f)| = |\int_0^1 f(t)\ dt| \leq \int_0^1 ||f||\ dt = ||f||$$

so $C = 1$.

Exercise. On $C([0,1])$ consider the norm

$$||f|| = \int_0^1 |f(x)|\, dx$$

given by the Riemann integral. Prove that the functional

$$\ell(f) = f(0),$$

is not bounded.

Solution: For every natural number n, let $f_n \in C([0,1])$ be a continuous function such that $f(0) = n$, $f(1/n) = 0$, $f(1) = 0$ and f is linear between these points. Then $||f_n|| = 1/2$ for all n, so this is a bounded sequence On the other hand $\ell(f_n) = n$ so the image is unbounded.