MATH 5210, LECTURE 3 - LEBESGUE IS COMPLETION OF RIEMANN
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Let $[0,1] \subset \mathbb{R}$, and $C([0,1])$ the space of continuous functions on $[0,1]$. On this space we have a norm

$$||f|| = \int_0^1 |f(x)|\, dx$$

given by the Riemann integral. Since the Riemann and Lebesgue integrals of continuous functions on $[0,1]$ coincide, the natural inclusion $C([0,1]) \subset L^1([0,1])$ is norm preserving. Hence $C([0,1])$ is a metric subspace of $L^1([0,1])$. In this lecture we shall prove that $L^1([0,1])$ is isomorphic to the completion of $C([0,1])$.

Let (X,d) be a metric space. Recall that the completion of X is the set X^* of equivalence classes of Cauchy sequences (x_n) in X. More precisely, given two Cauchy sequences (x_n) and (y_n), the sequence of distances $d(x_n, y_n)$ is a Cauchy sequence of real numbers, hence it has a limit $\lim_n d(x_n, y_n)$. This limit is the (pseudo) distance between two Cauchy sequences. Two Cauchy sequences are equivalent if their distance is 0. Now assume that X is a subset of a complete metric space Y. Then any Cauchy sequence (x_n) in X has a limit $\lim_n x_n \in Y$. Observe that any two equivalence sequences have the same limit. Thus we have a natural map

$$i : X^* \to Y$$

that sends any Cauchy sequence in X to its limit. Moreover, if X is dense, then this map is an isomorphism of X^* and Y, see HW 3 exercise on my web page.

Examples: Let $X = (0,1]$, with metric given by the usual distance between real numbers. The space X is not complete, since $(x_n) = (\frac{1}{n})$ is a Cauchy sequence in X, without a limit. Let $Y = [0,1]$. Then Y is compact and hence complete. Cleary X is a dense set in Y, therefore the completion of X is Y. The completion is a general abstract construction, however, sometimes it has a simple realization as in this example. A significantly more difficult example is $X = \mathbb{Q}$ and $Y = \mathbb{R}$. Hence $\mathbb{R}$ is isomorphic to the completion of $\mathbb{Q}$. Of course, $\mathbb{R}$ is sometimes defined as such, but there is another, wonderful, definition of $\mathbb{R}$ by Dedekind cuts. In particular, the two definitions are equivalent.

Let's go back to $C([0,1]) \subset L^1([0,1])$. We know that $L^1([0,1])$ is complete, so it remains to show that $C([0,1])$ is dense in $L^1([0,1])$, that is, for every $f \in L^1([0,1])$

there exists $g \in C([0,1])$ such that

$$||f-g|| = \int |f-g| < \epsilon.$$

The proof of that is a trivial series of reductions involving what we already know. From the definition of the Lebesgue integral, there exists a simple integrable function φ such that

$$||f-\varphi|| = \int |f-\varphi| < \epsilon.$$

Recall that $\varphi = \sum_{i=1}^{\infty} c_i \chi_{A_i}$ where $A_i \subset [a,b]$ are Lebesgue measurable sets, χ_{A_i} is the characteristic function of A_i, and $c_i \in \mathbb{R}$. Since φ is integrable, for every $\epsilon > 0$, there exists n such that

$$\sum_{i>n} |c_i| \mu(A_i) < \epsilon.$$

This implies that $||\varphi - \varphi_n|| < \epsilon$, where $\varphi_n = \sum_{i=1}^{n} c_i \chi_{A_i}$. Thus f can be approximated arbitrarily close by finite linear combinations of χ_A, for measurable sets A. But, given a measurable set A, for any $\epsilon > 0$ there exists an elementary set E such that $\mu(A\Delta E) < \epsilon$, hence

$$\int |\chi_E - \chi_A| = \mu(A\Delta E) < \epsilon$$

Thus, since any elementary set is a disjoint union of intervals, it follows that f can be approximated arbitrarily close by finite linear combinations of characteristic functions of intervals. Hence it remains to do the following exercise:

Exercise: Show that the characteristic function of an interval, say $[a,b] \subset [0,1]$ can be approximated in $L^1([0,1])$ by continuous functions.

Solution: For every natural number n, let f_n be a continuous piece-wise linear function such that $f(x) = 0$ on $[0, a-1/n]$, slope n on $[a-1/n, a]$, $f(x) = 1$ on $[a,b]$, slope $-n$ on $[b, b+1/n]$ and $f(x) = 0$ on $[b+1/n, 1]$. Then

$$||\chi_{[a,b]} - f_n|| = 1/n.$$