Let \( X = \{2, 3, 5, \ldots\} \) denote the set of all prime numbers. Let \( A \subset X \). Recall, from the last lecture, that \( A \) has polar density \( \frac{m}{d} \) if \( \zeta_{Q,A}(s)^d \) has a pole of order \( m \) at \( s = 1 \). More precisely, this means
\[
\lim_{s \to 1^+} (s - 1)^m \zeta_{Q,A}(s)^d = c \neq 0.
\]
(Taking limit \( s \to 1^+ \) keeps us in the half plane \( \Re(s) > 1 \) where \( \zeta_{Q,S}(s) \) is absolutely convergent, so we don’t have to worry about analytic continuation.) Since \( \zeta_Q(s) \) has a simple pole at \( s = 1 \) with residue 1, the above can be rewritten as
\[
\lim_{s \to 1^+} \zeta_Q(s)^{-m} \zeta_{Q,A}(s)^d = c \neq 0
\]
and this implies (check it) that
\[
\lim_{s \to 1^+} \frac{\log \zeta_{Q,A}(s)}{\log \zeta_Q(s)} = \frac{m}{d}.
\]

We shall relate polar density to more commonly used Dirichlet’s density. The Dirichlet’s density of \( A \) is the limit
\[
\delta(A) := \lim_{s \to 1^+} \frac{\sum_{p \in A} \frac{1}{p^s}}{\sum_{p \in X} \frac{1}{p^s}}
\]
if it exists. We need the following lemma:

**Lemma 0.1.** Recall that \( X \) is the set of all prime numbers. Then
\[
\lim_{s \to 1^+} \frac{\sum_{p \in X} \frac{1}{p^s}}{\log \zeta_Q(s)} = 1
\]

**Proof.** Using Euler factorization,
\[
\log \zeta_Q(s) = \sum_{p \in X} - \log(1 - \frac{1}{p^s}).
\]
For \( 0 < x < 1 \) we have
\[
- \log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}
\]
thus
\[
\log \zeta_Q(s) = \sum_{p \in X} \frac{1}{p^s} + e(s).
\]
We now estimate the term \( e(s) \) using
\[
\sum_{n=2}^{\infty} \frac{x^n}{n} \leq \frac{1}{2} \sum_{n=2}^{\infty} x^n = \frac{1}{2} x^2 \frac{1}{1-x}.
\]

Furthermore, since
\[
\frac{1}{1 - \frac{1}{p^s}} \leq 2
\]
for \( s > 1 \) it follows that
\[
e(s) \leq \sum_{p \in \mathcal{X}} \frac{1}{p^{2s}} \leq \zeta_Q(2s).
\]
Thus \( \lim_{s \to 1^+} (e(s)/\log \zeta_Q(s)) = 0. \)

Exercise: If \( \log \zeta_{Q,A}(s) \to +\infty \) as \( s \to 1^+ \) then
\[
\lim_{s \to 1^+} \frac{\sum_{p \in A} \frac{1}{p^s}}{\log \zeta_{Q,A}(s)} = 1
\]

Thus, if a set \( A \) has a positive polar density, then \( \log \zeta_{Q,A}(s) \to +\infty \) as \( s \to 1^+ \), it follows at once from the lemma and the exercise that \( A \) has Dirichlet’s density equal to the polar density. In particular, from the previous lecture:

**Corollary 0.2.** The set of primes that split completely in a Galois extension of degree \( n \) has Dirichlet density \( 1/n \).

This is a great result, and a special case of the Čebotarev density theorem, that we shall discuss later.

Exercise: If \( A \) and \( B \) are two disjoint sets with Dirichlet measures prove that
\[
\delta(A \cup B) = \delta(A) + \delta(B).
\]

In words, Dirichlet measure is a finitely additive measure on \( X \). Is it countably additive?

Exercise: Assume \( F \) is a quadratic field, and \( B \) the set of primes that are inert (stay prime) in \( F \). Prove that \( \delta(B) = 1/2 \). As a consequence, the set of primes \( p \equiv 1 \pmod{4} \) and the set of primes \( p \equiv 3 \pmod{4} \) both have Dirichlet’s density 1/2, why?