MATH 6370, LECTURE 2
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Let $X = \{2, 3, 5, \ldots\}$ denote the set of all prime numbers. Let $A \subset X$. Recall, from the last lecture, that A has polar density $\frac{m}{d}$ if $\zeta_{\mathbb{Q},A}(s)^d$ has a pole of order m at $s = 1$. More precisely, this means

$$\lim_{s \to 1^+} (s-1)^m \zeta_{\mathbb{Q},A}(s)^d = c \neq 0.$$

(Taking limit $s \to 1^+$ keeps us in the half plane $\Re(s) > 1$ where $\zeta_{\mathbb{Q},S}(s)$ is absolutely convergent, so we don't have to worry about analytic continuation.) Since $\zeta_{\mathbb{Q}}(s)$ has a simple pole at $s = 1$ with residue 1, the above can be rewritten as

$$\lim_{s \to 1^+} \zeta_{\mathbb{Q}}(s)^{-m} \zeta_{\mathbb{Q},A}(s)^d = c \neq 0$$

and this implies (check it) that

$$\lim_{s \to 1^+} \frac{\log \zeta_{\mathbb{Q},A}(s)}{\log \zeta_{\mathbb{Q}}(s)} = \frac{m}{d}.$$

We shall relate polar density to more commonly used Dirichlet's density. The Dirichlet's density of A is the limit

$$\delta(A) := \lim_{s \to 1^+} \frac{\sum_{p \in A} \frac{1}{p^s}}{\sum_{p \in X} \frac{1}{p^s}}$$

if it exists. We need the following lemma:

Lemma 0.1. *Recall that X is the set of all prime numbers. Then*

$$\lim_{s \to 1^+} \frac{\sum_{p \in X} \frac{1}{p^s}}{\log \zeta_{\mathbb{Q}}(s)} = 1$$

Proof. Using Euler factorization,

$$\log \zeta_{\mathbb{Q}}(s) = \sum_{p \in X} -\log(1 - \frac{1}{p^s}).$$

For $0 < x < 1$ we have

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

thus

$$\log \zeta_{\mathbb{Q}}(s) = \sum_{p \in X} \frac{1}{p^s} + e(s).$$

We now estimate the term $e(s)$ using

$$\sum_{n=2}^{\infty}\frac{x^n}{n}\leq\frac{1}{2}\sum_{n=2}^{\infty}x^n=\frac{1}{2}x^2\frac{1}{1-x}.$$

Furthermore, since

$$\frac{1}{1-\frac{1}{p^s}}\leq 2$$

for $s>1$ it follows that

$$e(s)\leq\sum_{p\in X}\frac{1}{p^{2s}}\leq\zeta_{\mathbb{Q}}(2s).$$

Thus $\lim_{s\to 1^+}(e(s)/\log\zeta_{\mathbb{Q}}(s))=0$.

□

Exercise: If $\log\zeta_{\mathbb{Q},A}(s)\to+\infty$ as $s\to 1^+$ then

$$\lim_{s\to 1^+}\frac{\sum_{p\in A}\frac{1}{p^s}}{\log\zeta_{\mathbb{Q},A}(s)}=1$$

Thus, if a set A has a positive polar density, then $\log\zeta_{\mathbb{Q},A}(s)\to+\infty$ as $s\to 1^+$, it follows at once from the lemma and the exercise that A has Dirichlet's density equal to the polar density. In particular, from the previous lecture:

Corollary 0.2. *The set of primes that split completely in a Galois extension of degree n has Dirichlet density $1/n$.*

This is a great result, and a special case of the Čebotarev density theorem, that we shall discuss later.

Exercise: If A and B are two disjoint sets with Dirichlet measures prove that

$$\delta(A\cup B)=\delta(A)+\delta(B).$$

In words, Dirichlet measure is a finitely additive measure on X. Is it countably additive?

Exercise: Assume F is a quadratic field, and B the set of primes that are inert (stay prime) in F. Prove that $\delta(B)=1/2$. As a consequence, the set of primes $p\equiv 1\pmod 4$ and the set of primes $p\equiv 3\pmod 4$ both have Dirichlet's density $1/2$, why?