Let $V$ be a vector space over $\mathbb{R}$ and $\| \cdot \|$ a norm on $V$. Then $d(x, y) = \|x - y\|$ is a metric on $V$. Let $\sum_{i=1}^{\infty} v_i$ be a series, where $v_i \in V$. The series is absolutely convergent if

$$\sum_{i=1}^{\infty} \| v_i \| < \infty$$

Recall that $V$ is complete if every Cauchy sequence in $V$ is convergent. As we proved in class, instead of working with sequences, in order to prove that $V$ is complete, it suffices to prove that absolutely convergent series are convergent.

Let $X = [0, 1]^k \subset \mathbb{R}^k$ or more generally any box in $\mathbb{R}^k$. Let $L^1(X)$ be the space of Lebesque integrable functions on $X$. More precisely, $L^1(X)$ is the set of equivalence classes of integrable functions where $f$ is equivalent to $g$ if

$$\int |f - g| = 0.$$

This is the same as saying that $f = g$ almost everywhere i.e. except on the set of Lebesgue measure 0. We shall now prove that $L^1(X)$ is a complete normed space for the norm

$$\| f \| = \int |f|.$$

This result is an easy combination of the Monotone Convergence and Lebesgue Dominated Convergence Theorems. Let $\sum_{i=1}^{\infty} f_i$ be an absolutely convergent series of functions $f_i \in L^1(X)$ i.e.

$$\sum_{i=1}^{\infty} \int |f_i| = \sum_{i=1}^{\infty} \| f_i \| < \infty$$

We need to find a function $f \in L^1(X)$ to which this series converges. Consider the sequence of non-negative functions

$$\varphi_n = \sum_{i=1}^{n} |f_i|.$$

The sequence $(\varphi_n)$ is clearly monotone and

$$\int \varphi_n = \sum_{i=1}^{n} \int |f_i| = \sum_{i=1}^{n} \| f_i \| < \sum_{i=1}^{\infty} \| f_i \|$$
for every \( n \). Hence, by the Monotone Convergence Theorem, there exists an integrable function \( \varphi \), such that \( \lim_{n} \varphi_{n}(x) = \varphi(x) \) for almost all \( x \). i.e. except perhaps on a measure 0 set. Thus, for almost all \( x \), the series of real numbers

\[
\sum_{i=1}^{\infty} |f_{i}(x)| = \varphi(x)
\]

is convergent. In particular, for those \( x \), the series \( \sum_{i=1}^{\infty} f_{i}(x) \) is also convergent, and we define

\[
f(x) := \sum_{i=1}^{\infty} f_{i}(x).
\]

For other \( x \) (in the set of measure 0) we can set \( f(x) = 0 \) or \( f(x) = 1 \) or any other value. Different choice give different functions, but the same element in \( L^{1}(X) \).

Exercise: Explain why \( f \) is integrable.

Solution: \( f \) is a pointwise limit of the sequence \( \sum_{i=1}^{n} f_{i} \) of measurable functions so it is measurable. Any measurable function \( f \), such that \( |f| \) is bounded by an integrable function \( \varphi \), is integrable.

It remains to prove that the series \( \sum_{i=1}^{\infty} f_{i} \) converges to \( f \) in \( L^{1}(X) \), that is, for the sequence of partial sums

\[
g_{n} = \sum_{i=1}^{n} f_{i}
\]

we want to prove that

\[
\lim_{n} \|g_{n} - f\| = \lim_{n} \int |g_{n} - f| = 0.
\]

The sequence of functions \( |g_{n} - f| \) converges to 0 pointwise, so we need to justify that we can switch the order of the limit and integral. Observe that, from the triangle inequality,

\[
|g_{n}(x) - f(x)| \leq \sum_{i>n} |f_{i}(x)| \leq \varphi(x).
\]

Thus the sequence of functions \( |g_{n} - f| \) is bounded (dominated) by the integrable function \( \varphi \). Now by the Lebesgue Dominated Convergence Theorem, we can switch the order of limit and integral,

\[
\lim_{n} \int |g_{n} - f| = \int \lim_{n} |g_{n} - f| = \int 0 = 0.
\]

Thus we have proved the following:

**Theorem 0.1.** \( L^{1}(X) \) is a complete normed space.

Complete normed spaces are also called Banach spaces.