MATH 5210, LECTURE 1 - NORMED SPACES
MARCH 18

Let V be a real vector space. A norm on V is a function $||\cdot|| : V \to \mathbb{R}$ such that, for all $x, y \in V$ and $\lambda \in \mathbb{R}$,

- $||x+y|| \leq ||x|| + ||y||$
- $||\lambda \cdot x|| = |\lambda| \cdot ||x||$
- $||x|| \geq 0$, $||x|| = 0$ if and only if $x = 0$.

The norm defines a distance function $d(x,y) = ||x-y||$ on V turning V into a metric space.

Examples: $V = \mathbb{R}^k$, the set of all k-tuples $x = (x_1, \ldots, x_k)$ of real numbers. Here we have several norms, among them

$$||x||_\infty = \sup_i |x_i| \text{ and } ||x||_1 = |x_1| + \cdots + |x_k|.$$

These two norms are related by the following inequalities

$$||x||_\infty \leq ||x||_1 \leq k \cdot ||x||_\infty.$$

Let d_∞ and d_1 be the corresponding metrics. The above inequalities imply a relationship between balls

$$B_\infty(x,\epsilon) \subseteq B_1(x,\epsilon) \subseteq B_\infty(x, k\cdot\epsilon)$$

so topologies defined by two metrics on $\mathbb{R}^k$ conicide. In terms of Cauchy sequence this can be expressed as the following

Exercise: Let (x_n) be a sequence in $\mathbb{R}^k$. Then (x_n) is Cauchy for the distance d_1 if and only if it is a Cauchy for d_∞.

Solution: Assume (x_n) is Cauchy for d_∞. Then for $\epsilon > 0$ there exists N such that $d_\infty(x_n, x_m) < \epsilon/k$ for all $n, m > N$. Then $d_1(x_n, x_m) < \epsilon$ hence (x_n) is Cauchy for d_1. The other direction is trivial.

Situation is different and more interesting if we consider an infinite dimensional analogue of $\mathbb{R}^k$. Fix a segment $[a,b] \subset \mathbb{R}$. Let $V = C([a,b])$ be the space of continuous functions on $[a,b]$. Analogues of the two norms on $\mathbb{R}^k$ are

$$||f||_\infty = \sup_{x\in[a,b]} |f(x)| \text{ and } ||f||_1 = \int_a^b |f(x)|\, dx$$

where the latter is the Riemann integral of $f \in V$. We have seen that V is a complete metric space with respect to the sup-norm (d_∞). Indeed if (f_n) is a Cauchy sequence in V for the sup-norm, then for every $x \in [a,b]$ the sequence $(f_n(x))$ of real numbers is Cauchy, hence $f(x) = \lim_n f_n(x)$ is a well defined function. We have shown that f is continuous, in fact the limit of the sequence. Hence V is complete with respect to this metric. On the other hand, V is not complete with respect to the second metric, since we have given examples of non-convergent Cauchy sequences, one as a homework problem. We have also show that any metric space can be completed, and in the case at hand, the completion of V is the space of Lebesgue integrable functions, denoted by $L^1([a,b])$, and we are now working towards proving this fact.

More precisely, let f be a Lebesgue integrable function on $[a,b]$. Then we can define

$$||f|| = \int_{[a,b]} |f|$$

here $\int$ denote the Lebesgue integral. A trouble here is that $||f||$ could be 0, even if f is not 0, for example, f could be the characteristic function of any countable subset of $[a,b]$. In fact, from our homework problem, we know that $||f|| = 0$ if and only if $f = 0$ almost everywhere i.e. except on a set of measure 0. So typically $L^1([a,b])$ is defined as the set of equivalence classes of integrable functions where f and g are equivalent if $f - g = 0$ almost everywhere. Thus, $L^1([a,b])$ is not a space of functions, strictly speaking, but it is still a vector space. We shall give a more elegant and abstract treatment of this:

Let V be a real vector space. A *pseudo-norm* on V is a function $||\cdot|| : V \to \mathbb{R}$ such that, for all $x, y \in V$ and $\lambda \in \mathbb{R}$,

- $||x + y|| \leq ||x|| + ||y||$
- $||\lambda \cdot x|| = |\lambda| \cdot ||x||$
- $||x|| \geq 0$, $||0|| = 0$

In other words, similar to norm, except there could be non-zero x with $||x|| = 0$. Let W be the subset of all $x \in V$ such that $||x|| = 0$. Then W is a subspace of V. Indeed, if $||x|| = ||y|| = 0$ then $||x + y|| \leq 0$ by the first bullet, and $||x + y|| \geq 0$ by the third, so $||x + y|| = 0$. If $||x|| = 0$, then $||\lambda \cdot x|| = 0$ be the second bullet. Hence W is a subspace.

Examples:

- $V = \mathbb{R}^2$, $||(x,y)|| = |x|$. Then W is y-axis.
- V, the space of Lebesgue integrable funtions on $[a,b]$. Then W is the subspace of functions equal 0 almost everywhere. Check that this is a vector space.

Given a vector space V and a subspace W we can define an equivalence relation on V by $x \sim y$ if $x - y \in W$. Check that this is an equivalence relation. For every $x \in V$, let $[x]$ be the equivalence class of x,

$$[x] = \{y \in V \mid x \sim y \text{ i.e. } x - y \in W\}.$$

Let V/W denote the set of equivalence classes. In the first example above, equivalence classes are lines parallel to the y-axis. The upshot here is that V/W is naturally a vector space, called the quotient or factor space. The operations of addition and scalar multiplications on V/W are "inherited" from those on V:

$$[x] + [y] := [x + y] \text{ and } \lambda[x] := [\lambda x].$$

However, we need to check that these definitions do not depend on the choices of representatives of classes. For example if $z \in V$ such that $[x] = [z]$, we need to check that $[\lambda x] = [\lambda z]$. Indeed, if $[x] = [z]$, then $x \sim z$, i.e. $x - z \in W$. Then $\lambda(x - z) = \lambda x - \lambda z \in W$, since W is a subspace. Hence $\lambda x \sim \lambda z$, i.e. $[\lambda x] = [\lambda z]$.

Exercise: Show that the addition is well defined.

Solution: If $[x] = [z]$ then $x - z \in W$. Then $(x + y) - (z + y) = x - z \in W$, so $[x + y] = [z + y]$.

Remark: Passing to the set of equivalence classes is done in mathematics at all levels, often implicitly. For example, rational numbers are classes of equivalent fractions. We use fractions to perform addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

but the outcome is independent of the choices made. For example, $1/2$ and $2/4$ are equivalent factions, representing the same rational number. We have $1/2 + 1/3 = 5/6$ and $2/4 + 1/3 = 10/12$, the outcomes are different but equivalent fractions. This independence of choices is critical but hardly ever metioned, let alone proved.

It is not difficult to see that V/W is a vector space with respect to these operations. Indeed, all axioms of vector space follow from those for V. In particular the zero element in V/W is $[0]$, the class of $0 \in V$, and note that $[0] = W$. Going back to the situation when V is equipped with a pseudo norm and W is the subspace of all $x \in V$ such that $||x|| = 0$, then V/W becomes naturally a normed space using the definition

$$||[x]|| = ||x||,$$

Indeed, $||[x]|| = 0$ if and only if $||x|| = 0$ and this precisely means that $x \in W$ i.e. $[x] = [0]$. In the example when V/W is the set of lines parallel to the y-axis, the norm of a line is its distance to the y-axis.

Exercise: Show that the norm $||[x]||$ is well defined: If $[x] = [y]$ then $||x|| = ||y||$.

Solution: If $[x] = [y]$ then $x - y \in W$, that is $||x - y|| = 0$. By the triangular inequality,

$$||x|| \leq ||x - y|| + ||y|| = ||y||.$$

Similarly, $||y|| \leq ||x||$, so $||x|| = ||y||$.