MATH 5210, LECTURE 11 - ORTHOGONAL DECOMPOSITION APRIL 10

Let V be a Hilbert space and $W \subset V$ a closed subspace. The orthogonal complement of W is the set

$$W^{\perp} = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

Exercise: Prove that $W^{\perp}$ is a closed subspace of V.

Solution. Let v be a limit point of $W^{\perp}$, that is, $v = \lim_n v_n$ where (v_n) is a sequence in $W^{\perp}$. Let $w \in W$. Then, by a previous exercise,

$$(v, w) = \lim_n (v_n, w) = \lim_n 0 = 0.$$

Hence $v \in W^{\perp}$, so $W^{\perp}$ is closed.

If v is contained in W and $W^{\perp}$ then $(v, v) = 0$ hence $v = 0$. Thus $W \cap W^{\perp} = 0$. Terminology complement comes from the following:

Proposition 0.1. *Any $x \in V$ can be uniquely written as a sum $x = y + z$ where $y \in W$ and $z \in W^{\perp}$.*

Proof. The idea of the proof is that y is the element in W closest to x. That is, y minimizes the function

$$f(w) = ||x - w||^2.$$

where $w \in W$. Assume that $y \in W$ is the minimum of the function f. Fix $w \in W$. Let $t \in \mathbb{R}$. Then the function of t

$$f(y+tw) = ||x-(y+tw)||^2 = (x-y-tw, x-y-tw) = (x-y, x-y) - 2t(x-y, w) + t^2(w, w)$$

has the minimum at $t = 0$. Thus $f'(0) = 0$ which works out to $(x - y, w) = 0$. This is true for all $w \in W$, hence $z = x - y \in W^{\perp}$. So we need to show that there exists the closest y. Since f is non-negative $\delta = \inf_w f(w)$ exists. Let y_n be a sequence in W such that $\lim_n f(y_n) = \delta$. If (y_n) is a Cauchy sequence, then $y = \lim_n y_n$. This limit exists since V is complete, and it is contained in W since W is closed. To prove that (y_n) is Cauchy, we need the parallelogram identity (check it):

$$||v + u||^2 + ||u - v||^2 = 2||v||^2 + 2||u||^2$$

Put $v = x - y_n$ and $u = x - y_m$,

$$||2x - (y_n + y_m)||^2 + ||y_n - y_m||^2 = 2||x - y_n||^2 + 2||x - y_m||^2.$$

Observe that $(y_n + y_m)/2 \in W$ and

$$||2x - (y_n + y_m)||^2 = 4||x - (y_n + y_m)/2|| = 4f((y_n + y_m)/2) \geq 4\delta.$$

Thus the parallelogram identity yields

$$||y_n - y_m||^2 = 2f(y_n) + 2f(y_m) - 4f((y_n + y_m)/2)) \leq 2f(y_n) + 2f(y_m) - 4\delta.$$

As $n, m \to \infty$, $f(y_n), f(y_m) \to \delta$, hence $||y_n - y_m||^2 \to 0$, thus (y_n) is Cauchy.

Exercise: Prove uniqueness of the decomposition $x = y+z$. Hint: use that $W \cap W^\perp = 0$.

Solution: Let $x = y' + z'$ be another decomposition where $y' \in W$ and $z' \in W^\perp$. Then

$$0 = x - x = (y + z) - (y' + z') = (y - y') + (z - z').$$

Thus $y - y' = z' - z$. But $y - y' \in W$ and $z' - z \in W^\perp$. Thus $y - y' = z' - z = 0$ since $W \cap W^\perp = 0$. Hence $y = y'$ and $z = z'$.

□

Example: Let $V = L^2([-1, 1])$. Let W be the subspace of even functions. Then $W^\perp$ is the subspace of odd functions. (Check that W is closed.)

We derive two wonderful consequences of the proposition. We can define

$$P : V \to W$$

$P(x) = y$ where $x = y+z$ is the decomposition given in the proposition, for $x \in V$. It is trivial to check that this is linear transformation. Moreover, since $||x||^2 = ||y||^2 + ||z||^2$ for perpendicular y and z,

$$||P(x)|| = ||y|| \leq ||x||$$

the linear transformation P is continuous. Clearly $P(x) = x$ for $x \in W$ and $P(x) = 0$ for $x \in W^\perp$ so P is called orthogonal projection of V onto W.

The second consequence is classification of continuous linear functionals on V. Observe that any $y \in V$ defines a linear functional $\ell_y : V \to \mathbb{R}$ via the scalar product

$$\ell_y(x) = (x, y)$$

for all $x \in V$. This functional is bounded, hence continuous, since

$$|\ell_y(x)| = |(x, y))| \leq ||y|| \cdot ||x||$$

by the Cauchy-Schwarz inequality. Conversely:

Corollary 0.2. *Let $f : V \to \mathbb{R}$ be continuous functional. Then there exists $y \in V$ such that*

$$f(x) = (x, y)$$

for all $x \in V$.

Proof. If $f = 0$ then $y = 0$, so assume $f \neq 0$. Since f is continuous, its null-space $W = f^{-1}(0)$ is closed. Let $W^\perp$ be its orthogonal complement. We claim that $W^\perp$ is one-dimensional. If u, v are two non-zero elements in $W^\perp$, then their linear combination

$$f(v)u - f(u)v$$

is in $W^\perp$. On the other hand, evaluating f on this element,

$$f(f(v)u - f(u)v) = f(v)f(u) - f(u)f(v) = 0$$

so this element is also in W. Hence $f(v)u - f(u)v = 0$, i.e u and v are dependent. Let e span the line $W^\perp$ and we can assume that $(e, e) = 1$. Any element $v \in V$ can be written uniquely as $v = w + te$, for some $w \in W$ and $t \in \mathbb{R}$. Then

$$f(v) = f(w + te) = f(w) + tf(e) = tf(e) = (v, y)$$

where $y = f(e)e$ (check the last equality).

□