## ALGEBRA - LECTURE X

### 1. Characters

Let $G$ be a group. In order to avoid any confusion with 0 and 1, let $e$ denote the identity of $G$. A character of $G$ is a homomorphism

$$\chi : G \to \mathbb{C}^\times.$$

The set of all characters of $G$ is denoted by $\hat{G}$. This is a group with respect to natural multiplication of characters; If $\chi_1$ and $\chi_2$ are two characters then $\chi_1 \cdot \chi_2$ is defined by

$$(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$$

for every $g$ in $G$. However, the group may not have any non-trivial characters. For example, $G = SL_n(\mathbb{Z})$ is perfect if $n \geq 3$, so $\hat{G}$ is trivial.

The theory of characters is particularly rich when $G$ is commutative. Henceforth we shall assume that $G$ is commutative and finite. Since $g^{|G|} = e$ by the theorem of Lagrange,

$$1 = \chi(e) = \chi(g^{|G|}) = \chi(g)^{|G|}$$

and it follows that $\chi(g)$ is a root of 1. Consider now $G = \mathbb{Z}/n\mathbb{Z}$. Here $e = 0$ and the group is generated by 1. Hence any character $\chi$ is determined by its value $\chi(1)$ and this value is an $n$-th root of 1, not necessarily primitive. If we denote the group of all complex $n$-th roots of 1 by $\mu_n$, then $\chi \mapsto \chi(1)$ defines an isomorphism

$$\widehat{(\mathbb{Z}/n\mathbb{Z})} \cong \mu_n.$$

By the theory of elementary divisors any finite abelian group $G$ is isomorphic to

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}$$

for some positive integers $d_1 | \ldots | d_k$. It is clear that $\widehat{G_1 \times G_2} \cong \hat{G}_1 \times \hat{G}_2$. Thus, we have shown:

**Proposition 1.1.** *Let $G$ any finite abelian group. With notations as above, we have*

$$\hat{G} \cong \mu_{d_1} \oplus \cdots \oplus \mu_{d_k}.$$

Since $\mu_n$ is generated by a primitive $n$-th root of 1, it is cyclic and therefore isomorphic to $\mathbb{Z}/n\mathbb{Z}$. This shows that

$$G \cong \hat{G}$$

however, this isomorphism is <u>not</u> canonical.

Every $g$ in $G$ defines a character $i_g$ of $\hat{G}$ by

$$i_g(\chi) = \chi(g)$$

for every $\chi$ in $\hat{G}$. The map $g \mapsto i_g$ defines a canonical homomorphism $i$ from $G$ into $\hat{\hat{G}}$.

**Proposition 1.2.** *The map $i$ is a canonical isomorphism of $G$ and $\hat{\hat{G}}$.*

*Proof.* Since the two groups have the same order, it suffices to show that $i$ is injective. Let $g$ be in the kernel of $i$. Then for every character $\chi$

$$1 = i_g(\chi) = \chi(g).$$

This shows that all characters of $G$ factor down to the quotient group $G/\langle g\rangle$. Since the number of characters of a finite abelian group is equal to the order of the group, $g$ must be the identity element. The proposition is proved. □

## 2. Fourier Transform

In this section we study the space of complex valued functions on a finite abelian group $G$. There are two natural choices of $L^2$-norms here:

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\bar{g}(x)$$

and

$$\langle f, g\rangle' = \sum_{x\in G} f(x)\bar{g}(x)$$

where $f$ and $g$ are any two functions on $G$. The corresponding Hilbert spaces will be denoted by $L^2(G)$ and $L^2(G)'$.

**Proposition 2.1.** *Let $\chi$ and $\chi'$ be any two characters of $G$. Then*

$$\langle \chi, \chi'\rangle = \delta_{\chi,\chi'}.$$

*In particular, the characters form an orthonormal basis of $L^2(G)$.*

*Proof.* In the following identities we use that the complex conjugate of $\chi'(x)$ is equal to $\chi'(x^{-1})$. If $\chi = \chi'$ then

$$\langle \chi, \chi\rangle = \frac{1}{|G|}\sum_{x\in G}\chi(x)\chi(x^{-1}) = \frac{1}{|G|}\sum_{x\in G} 1 = 1.$$

If $\chi \neq \chi'$ then there exists $y$ in $G$ such that $\chi(y) \neq \chi'(y)$. As $x$ runs through all elements in $G$ so does $xy$. It follows that

$$\langle \chi, \chi'\rangle = \frac{1}{|G|}\sum_{x\in G}\chi(xy)\chi'(y^{-1}x^{-1}) = \chi(y)\chi'(y^{-1})\langle \chi, \chi'\rangle.$$

Since $\chi(y)\chi'(y^{-1}) \neq 1$ this identity is possible only if $\langle \chi, \chi'\rangle = 0$. The proposition is proved. □

For any $f$ in $L^2(G)$ define $\hat{f}$ in $L^2(\hat{G})'$ by

$$\hat{f}(\chi) = \langle f, \chi\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\chi(x^{-1}).$$

Here we used that $\bar{\chi}(x) = \chi(x^{-1})$ for any character $\chi$ of $G$. The map $f \mapsto \hat{f}$ is the Fourier transform on the group $G$. As an example, let us calculate the Fourier transform of the delta function $\delta_e$ of the identity element $e$ of $G$.

$$\hat{\delta}_e(\chi) = \frac{1}{|G|}\sum_{x\in G}\delta_e(x)\chi(x^{-1}) = \frac{1}{|G|}.$$

This shows that the Fourier transform is the constant function $\frac{1}{|G|}$.

**Theorem 2.2.** *The Fourier transform is an isometry between $L^2(G)$ and $L^2(\hat{G})'$.*

*Proof.* Clearly, the Fourier transform is linear. We know that the characters form an orthonormal basis of $L^2(G)$. The Fourier transform of a character $\chi$ is, by definition,

$$\hat{\chi}(\chi') = \langle \chi, \chi' \rangle.$$

It follows that $\hat{\chi}$ is the delta function $\delta_\chi$. Since delta functions of individual characters form an orthonormal basis of $L^2(\hat{G})'$ we are done. □

**Corollary 2.3.** *(Plancherel formula) For any function $f$ on $G$ we have*

$$f(e) = \sum_{\chi \in \hat{G}} \hat{f}(\chi).$$

*Proof.* Note that $f(e) = |G| \cdot \langle f, \delta_e \rangle$. By the theorem, we have

$$\langle f, \delta_e \rangle = \langle \hat{f}, \hat{\delta}_e \rangle'.$$

As we have calculated above, the Fourier transform of $\delta_e$ is the constant function $\frac{1}{|G|}$. This completes the proof. □

## 3. Representation Theory

In this section $V$ is a finite dimensional vector space over $\mathbb{C}$. Let $G$ be a group (no condition on it, yet). A representation of $G$ is a pair $(\pi, V)$ where $\pi$ is a homomorphism

$$\pi : G \to GL(V).$$

In particular, representations are generalizations of characters. Characters are simply one dimensional representations.

A basic example is $G = S_n$, the symmetric group, $V = \mathbb{C}^n$, and the action of $S_n$ is by permuting coordinates of $\mathbb{C}^n$. In particular, $\pi(g)$ is an $n \times n$ permutation matrix.

Let $(\pi, V)$ be a representation of $G$. A subspace $U \subseteq V$ is an invariant subspace of $V$ if

$$\pi(g)U \subseteq U$$

for all $g$ in $G$. We also say that $U$ is a subrepresentation of $V$. A representation $V$ is irreducible if 0 and $V$ are only invariant subspaces of $V$.

For example, $S_n$ has two obvious invariant subspaces in $\mathbb{C}^n$. The first one is the line $\ell$ consisting of all $n$-tuples $(x, x, \ldots, x)$. The other one is the hyperplane $H$ consisting of all $n$-tuples $(x_1, x_2, \ldots, x_n)$ such that

$$x_1 + x_2 + \cdots + x_n = 0.$$

The space $H$ can be visualised in the case $n = 3$ as follows. The hyperplane $H$ is spanned by the following three linearly dependent vectors:

$$v_1 = (2, -1, -1), v_2 = (-1, 2, -1) \text{ and } v_3 = (-1, -1, 2).$$

Note that $v_1 + v_2 + v_3 = 0$. The three vectors have the same length and it is easy to check that the angle between any two of them is 120°. In particular, the three vectors represent vertices of a regular triangle. With this notation, $S_3$ acts on the triangle as a group of isometries.

Using $v_1$ and $v_2$ as a basis we can write $\pi(g)$ - the action of $g$ on $H$ - as a 2 by 2 matrix, for every $g$ in $S_3$. For example, let $g$ be the cyclic permutation (123). Then

$$\pi(g)v_1 = v_2 \text{ and } \pi(g)v_2 = v_3 = -v_1 - v_2.$$

Thus, with respect to the basis $v_1, v_2$, the matrix $\pi(g)$ is

$$\pi(g) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Let $(\pi, V)$ and $(\rho, U)$ be two representations of $G$. A linear map $T : V \to U$ is called an intertwining map if it commutes with the action of $G$. This means that for every $g$ in $G$

$$T(\pi(g)(v)) = \rho(g)(T(v))$$

for every $v$ in $V$. Note that $ker(T)$ and $im(T)$ are invariant subspaces of $V$ and $U$, respectively. In particular, if $V$ and $U$ are irreducible then either $T = 0$ or $T$ is a bijection. In this case $(\pi, V)$ and $(\rho, U)$ are said to be isomorphic.

**Proposition 3.1.** *(Schur's lemma) Let $(\pi, V)$ be an irreducible representation of $G$. Let $T : V \to V$ be an intertwining map. Then $T = \lambda \cdot 1_V$ for some complex number $\lambda$.*

*Proof.* Let $\lambda$ be an eigenvalue of $T$. Then $T_\lambda = T - \lambda \cdot 1_V$ is an intertwining map with a non-trivial kernel. Since $V$ is irreducible, the kernel of $T_\lambda$ must be equal to $V$. This shows that $T = \lambda \cdot 1_V$, as claimed. □

Let $V$ be a representation of $G$. We say that $V$ is semi-simple if for every invariant subspace $U$ of $V$, there exists a complementary invariant subspace $U'$. This means that that

$$V \cong U \oplus U'.$$

For example, if $G = S_n$ and $V = \mathbb{C}^n$, then $\ell$ is an invariant subspace and $H$ its invariant complement. However, not every representation is semi-simple. As an example, consider $G = \mathbb{Z}$ and let $(\pi, V)$ be a representation of $\mathbb{Z}$ defined as follows. Let $V = \mathbb{C}^2$, which we think of as the set of all 2 by 1 matrices. For every $z$ in $\mathbb{Z}$ let

$$\pi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

The line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the unique proper invariant subspace of $V$. In particular, it has no complement. Thus this representation is not semi-simple. However, if $G$ is finite then every representation is semi-simple (or one says that it decomposes completely):

**Proposition 3.2.** *Let $(\pi, V)$ be a representation of a finite group $G$. Let $U$ be an invariant subspace of $V$. Then there exists a complemantary invariant subspace $U'$:*

$$V \cong U \oplus U'.$$

*Proof.* Let $P : V \to U$ be a projector onto $U$. If $P$ intertwines the action of $G$ then $U' = ker(U)$ does the job. Otherwise, define

$$P' = \frac{1}{|G|} \sum_{x \in G} \pi(x^{-1}) P \pi(x).$$

We shall show that $P'$ is a projector onto $U$ which intertwines the action of $G$. First of all, for every $x$ in $G$,

$$\pi(x^{-1})P\pi(x)(U) \subseteq U$$

since $U$ is invariant by $\pi(x)$. This shows that $P'(U) \subseteq U$. Moreover, for every $v$ in $U$ we have

$$P'(v) = \frac{1}{|G|}\sum_{x\in G}\pi(x^{-1})P\pi(x)(v) = \frac{1}{|G|}\sum_{x\in G}\pi(x^{-1})\pi(x)(v) = v.$$

This shows that $P'$ is a projection onto $U$. It remains to show that $P'$ intertwines the action of $G$. Let $y$ be in $G$. Then

$$P' \circ \pi(y) = \frac{1}{|G|}\sum_{x\in G}\pi(x^{-1})P\pi(xy).$$

Now, as $x$ runs through all elements of $G$, so does $z = xy$. Since $x^{-1} = yz^{-1}$, it follows that

$$P' \circ \pi(y) = \frac{1}{|G|}\sum_{z\in G}\pi(yz^{-1})P\pi(z) = \pi(y) \circ P'.$$

This shows that $P'$ is an intertwining projector. The proposition is proved. □

## 4. Orthogonality of matrix coefficients

Let $(\pi, V)$ be a representation of $G$, not necessarily irreducible. Matrix coefficient of $V$ is a function on $G$ defined by

$$c_{v,v^*}(g) = v^*(\pi(g)v)$$

where $v$ is in $V$ and $v^*$ in $V^*$, the space of linear functionals on $V$. The name matrix coefficient comes from the following. Let $v_1, \ldots, v_n$ be a basis of $V$ and $v_1^*, \ldots, v_n^*$ the dual basis in $V^*$. This means that $v_j^*(v_i) = \delta_{ij}$. In terms of the basis $v_1, \ldots, v_n$ every operator $\pi(g)$ can be written as an $n \times n$ matrix. The entry on the intersection of the $i$'th column and $j$'th row is $c_{v_i,v_j^*}(g)$ or $c_{i,j}(g)$, abbreviated. Note that matrix coefficients of $V$ form a subspace of $\mathbb{C}[G]$, the space of functions on $G$. Moreover, any matrix coefficient of $V$ is a linear combination of the coefficients $c_{i,j}(g)$. In fact, since the map $(v, v^*) \mapsto c_{v,v^*}$ is bilinear in both variables, we have a natural map

$$C_V : V \otimes V^* \to \mathbb{C}[G].$$

Assume now that $(\pi, V)$ is <u>irreducible</u>. The main result of this section is that the matrix coefficients $c_{ij}(g)$ are linearly independent functions on $G$. Thus, the space of matrix coefficients of $V$ has dimension $n^2$ and the map $C_V$ is injective.

The approach to proving this result is based on the following simple lemma:

**Lemma 4.1.** *Let $E$ be a vector space and $(-,-)$ a bilinear form on $E$. Let $e_1, \ldots, e_m$ and $e_1^*, \ldots, e_m^*$ be two collections of vectors in $E$ such that $(e_i, e_j^*) = \delta_{ij}$. Then $e_1, \ldots, e_m$ are linearly independent.*

We apply this lemma to $E = \mathbb{C}[G]$ and

$$(f, h) = \frac{1}{|G|}\sum_{g\in G} f(g)h(g^{-1}).$$

**Proposition 4.2.** *Let $(\pi, V)$ and $(\rho, U)$ be two irreducible and non isomorphic representations of $G$. Then*

$$(c_{v,v^*}, c_{u,u^*}) = 0.$$

*Proof.* Fix $v^*$ in $V$ and $u$ in $U$. Define a map $P : V \to U$ by

$$P(v) = v^*(v) \cdot u.$$

Then

$$P' = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) P \pi(g)$$

is an intertwining map from $V$ to $U$. Since $V$ and $U$ are not isomorphic, $P'$ must be 0. This implies that for every $v$ in $V$ and $u^*$ in $U$ we have $P'(v) = 0$ and, therefore, $u^*(P'(v)) = 0$. But this is essentially what we wanted to prove. Indeed, note that

$$\rho(g^{-1})[P\pi(g)(v)] = \rho(g^{-1})[v^*(\pi(g)v) \cdot u] = v^*(\pi(g)v) \cdot \rho(g^{-1})u.$$

Evaluating at $u^*$ and summing up over all $g$ in $G$ gives the desired identity. The proposition is proved. □

If $V = U$ we have a similar result.

**Proposition 4.3.** *Let $(\pi, V)$ be an irreducible representation. Let $v_1, \ldots, v_n$ be a basis of $V$ and $v_1^*, \ldots, v_n^*$ the dual basis of $V^*$. Let $c_{i,j}(g)$ be the matrix coefficient corresponding to $v_i$ and $v_j^*$. Then*

$$(c_{i,j}, c_{k,l}) = \delta_{kj}\delta_{il} \cdot \frac{1}{n}.$$

*Proof.* Define a map $P_{jk} : V \to V$ by

$$P(v) = v_j^*(v) \cdot v_k.$$

Then $P$ is an elementary matrix, with 1 at the intersection of $j$-th column and $k$'th row. In particular

$$trace(P_{jk}) = \delta_{jk}.$$

Define

$$P'_{jk} = \frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) P_{jk} \pi(g).$$

It is an intertwining map on $V$. By Schur's lemma it is equal to $\lambda 1_V$. Note that

$$\lambda = \frac{1}{n} trace(P'_{jk}).$$

Since the trace of $P_{jk}$ is equal to the trace of a conjugate of $P_{jk}$, and $P'_{jk}$ is a sum of conjugates of $P_{jk}$, it follows that the trace of $P'_{jk}$ is equal to the trace of $P_{jk}$). This determines $\lambda$. It follows that

$$P'_{jk} v_i) = \frac{1}{n} \delta_{jk} v_i$$

and

$$v_l^*[P'_{jk}(v_i)] = \frac{1}{n} \delta_{jk} \delta_{il}.$$

□

This shows that $c_{ij}$ are dual to $c_{ji}$ with respect to the pairing $(\cdot,\cdot)$. In particular:

**Corollary 4.4.** *Let $V_1,\dots,V_r$ be irreducible, mutually non-isomorphic representations of $G$ of dimensions $n_1,\dots,n_r$. Then*

$$|G| \leq \sum_{i=1}^{r} n_i^2.$$

*In particular, the number of irredcible representations of $G$ is finite.*

## 5. Characters

Let $(\pi, V)$ be a representation of $G$. The character $\chi$ of $\pi$ is defined by

$$\chi(g) = trace(\pi(g)).$$

With respect to a basis $v_1,\dots,v_n$ of $V$ we clearly have

$$\chi(g) = \sum_{i=1}^{n} c_{ii}(g).$$

Since $g$ has finite order any eigenvalue of $\pi(g)$ is a root of 1. Since the trace of $\pi(g)$ is the sum of its eigenvalues, it follows that

$$\chi(g^{-1}) = \overline{\chi(g)}$$

where $\overline{\chi(g)}$ denotes the complex conjugate of $\chi(g)$. The orthogonality relations between matrix coefficients proved in the last section imply the orthogonality relations between characters of irreducible representations:

**Proposition 5.1.** *Let $V_1,\dots,V_r$ be irreducible mutually non-isomorphic representations of $G$. Then*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}.$$

If $V$ is not irreducible, then we can decompose it into irreducible summands:

$$V = n_1 V_1 \oplus \cdots \oplus n_r V_r$$

where $n_i$ is a non-negative integer, called the muliplicity of the irreducible representation $V_i$ in $V$. Since the trace of a block diagonal matrix is a sum of traces of the blocks, it follows that

$$\chi = m_1\chi_1 + \cdots + m_r\chi_r.$$

Orthogonality relations of characters imply that

$$\langle \chi, \chi \rangle = m_1^2 + \cdots + m_r^2 \text{ and } m_i = \langle \chi, \chi_i \rangle.$$

In particular, a representation $(\pi, V)$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$. As an example, consider our 2-dimensional representation of $S_3$. The trace function is constant on conjugacy classes, so it is easy to tabulate:

| $g$ | $\chi(g)$ |
|---|---|
| $1$ | $2$ |
| $(12),(23),(13)$ | $0$ |
| $(123),(132)$ | $-1$ |

It follows that $\langle \chi, \chi \rangle = \frac{1}{6}(2^2 + (-1)^2 + (-1)^2) = 1$ and $\pi$ is irreducible.

Right and left regular representations of $G$ are defined on the space of functions $\mathbb{C}[G]$ by

$$[R(g)f](x) = f(xg) \text{ and } [L(g)f](x) = f(g^{-1}x)$$

for every $f$ in $\mathbb{C}[G]$. In order to decompose the right regular representation we need to compute its trace. Bu this is easy! Indeed, the space $\mathbb{C}[G]$ is spanned by delta functions $\delta_y$ for all $y$ in $G$. Note that

$$R(g)(\delta_y) = \delta_{yg^{-1}}$$

This shows that in terms of the basis $\delta_y$ the trace of $R(g)$ is given by

$$\chi_R(g) = \begin{cases} |G| \text{ if } g = 1 \\ 0 \text{ if } g \neq 1. \end{cases}$$

Let $V_i$ be an irreducible representation of dimension $n_i$. Then $\chi_i(1) = n_i$ and

$$\langle \chi_R, \chi_i \rangle = n_i$$

since the sum reduces to $g = 1$.

**Theorem 5.2.** *Let $G$ be a finite group and $V_1, \ldots, V_r$ all its irreducible representations. Then*

$$\mathbb{C}[G] = n_1 V_1 \oplus \cdots \oplus n_r V_r.$$

*In particular $|G| = n_1^2 + \cdots + n_r^2$.*

The above theorem can be used to show that a list of representations of a group is complete. For example, $S_3$ has two one dimensional representations, the trivial and the sign character, and the 2 dimensional irreducible representation given by symmetries of an equilateral triangle. The sum of squares of their dimensions ads up to 6 the order of the group. In particular the three representations exhaust all irreducible representations of $S_3$.

## HW problems

1) Let $D_8$ be the group of symmetries of a squares. Show that the 2 dimensional representation (as symmetries of the square) is irreducible by calculating the character table.

2) Write down the character table of all 1-dimensional representations of $D_8$. Hint: let $-1$ in $D_8$ be the symmetry of the square given by multiplication by the real number $-1$ (the rotation by $180^\circ$). Consider the characters of the quotient of $D_8$ by $\{1, -1\}$.

3) Let $V$ be an irreducible subspace of the right regular representation of $G$ on $\mathbb{C}[G]$. Show that $V$ is contained in the space of its matrix coefficients. Hint: pick $v^*$ to be the functional given by evaluating a function at 1.