

SLCSE Math 1050, Spring, 2013
Lesson 1, Monday, January 7, 2013: Quadratic Functions: Solutions

Problem1. A commuter train carries 600 passengers each day from a suburb to a city. It costs \$ 1.50 per person to ride the train. It is found that 40 more people will ride the train for each 5 cent decrease in the fare. What fare should be charged to make the largest possible revenue?

Solution. Here the variables are x , the number of passengers, p , the price per ticket, and R , the revenue. We have $R = px$, and the given information provides a relation between p and x . In order to most easily see this relation, we introduce a new variable: n , the number of 5 cent decreases. Then the price is $p = 150 - 5n$ cents, and the number of passengers is $x = 600 + 40n$. We obtain

$$R = (150 - 5n)(600 + 40n) .$$

To find the value of n which maximizes revenue, we complete the square to find the vertex of the parabola. First, multiply out:

$$\begin{aligned} R &= 9000 - 3000n + 6000n - 200n^2 = 9000 - 200(n^2 - 15n) \\ &= 9000 - 200(n - 7.5)^2 + 200\left(\frac{225}{4}\right) . \end{aligned}$$

Thus $n = 7.5$ at the vertex (for the maximum), and the price which maximizes the revenue is $p = 150 - 5(7.5) = 150 - 37.5 = 112.5$ cents. We also read off that the maximum revenue is

$$R_{max} = 9000 + 50(225) = 20,250 .$$

Problem 2. An automobile manufacturer sells, on average, 8000 cars per month at twenty-five thousand dollars. The marketing department has determined that for every one thousand dollar reduction in price, the company can sell an additional 500 cars per month. At what price should the car be sold so as to maximize revenue?

Solution. Let p be the price reduction (in thousands of dollars) at which the automobile is to be sold. With this reduction, the manufacturer sells $8 + p/2$ thousand cars, and the price is $25 - p$ thousand. Thus the total revenue is $R = (8 + p/2)(25 - p)$. To find the maximum, we multiply out and complete the square:

$$R = 200 + 4.5p - p^2/2 = 200 - \frac{1}{2}(p^2 - 9p + \quad) .$$

Actually, since we are asked for the selling price to maximize revenue, we already know that the square term is going to be $(p - 4.5)^2$, and so the maximum is achieved at a reduction of $p = \$4500$ and the selling price should be \$20,500.

Problem 3. The Jones Jumpersuit Company can sell $400 - 8p$ jumpersuits each month at a price of $120 + p$ dollars. Jones has fixed costs of \$ 8000 per month, and the cost in labor and material for each suit is \$ 25. At what price will Jones maximize profit?

Solution. The cost of producing $400 - 8p$ jumpersuits is

$$\text{Cost} = 8000 + 25(400 - 8p)$$

and the income from this sale is

$$\text{Income} = (400 - 8p)p .$$

Profit is

$$\text{Profit} = \text{Income} - \text{Cost} = (400 - 8p)p - (8000 + 25(400 - 8p)) = -8p^2 + 600p - 18000 .$$

Now start completing the square:

$$P = -8(p^2 - 75p + \quad) - 18000 ,$$

so the profit is maximized at $p = \$ 37.50$.

Problem 4. Tickets for the Giulia Opera sell at \$65 each. However, for groups, this deal is offered: for every 10 tickets over 100 purchased there will be a 2% discount on all tickets. Of course, as we can see there has to be an upper limit: for a group of 600 people, the tickets will be free. Giulia decides to set a limit at that number which maximizes the total revenue. What is that number?

Solution. Let's let x be the number of increments of ten-ticket sales that are actually sold; that is, we sell $100 + 10x$ tickets. Then the per-ticket price is $(1 - 0.02x)65$ and the total revenue is

$$R = (\text{price}) \times (\text{number sold}) = (100 + 10x)(1 - 0.02x)65 .$$

Multiply this out:

$$R = 100 - 2x + 10x - .2x^2 = 100 + 8x - .2x^2 = 100 - .2(x^2 - 40x) .$$

By now, we know that we'll get $R = \text{something} - .2(x - 20)^2$, and so the maximum is attained when $x = 20$, and the number of ticket sales maximizing the revenue is $100 + 10(20) = 300$.

Problem 5. a) Graph a) $y = x + C/x$ for x positive. Notice that as $x \rightarrow 0$ or $x \rightarrow \infty$, y becomes infinite.

b) Find the coordinates of the minimum point.

Solution. Multiply $y = x + C/x$ by x and rewrite as $x^2 - yx + C = 0$, and look at the graph again. For a given y , the x coordinate of the points on the graph are the roots of this quadratic equation. Thus the minimum point for the graph is where there is only one x for the value of y that is, where the quadratic equation has only one real root. By the discriminant condition, this is $y^2 - 4C = 0$, or $y = 2\sqrt{C}$. Now we can solve for x to get $x = \sqrt{C}$. Thus the minimum point is $(\sqrt{C}, 2\sqrt{C})$.

Problem 6. Find the dimensions of the right triangle with one vertex at the origin, another on the positive x -axis, and the third on the curve $y = 4 + x^{-2}$ which is of minimum area.

Solution. The variables are: area A , the length of the base x , and the height y of the triangle. The variables are related by

$$A = \frac{1}{2}xy , \quad y = 4 + x^{-2} ,$$

so $A = (1/2)x(4 + x^{-2}) = 2x + (1/2)x^{-1}$. Rewrite this so that we can use the answer to problem 6:

$$A = 2\left(x + \frac{1/4}{x}\right) ,$$

so the C of problem 6 is $1/4$, and thus the minimum point of the graph occurs when $x = 1/2$, $y = 8$, and the area is $A = 2$.

Problem 7. Farmer Brown wants to build a right triangular chicken coop containing 100 square feet. The hypotenuse will lie on an existing wall, but the other two sides are to be built. What should the dimensions of these sides be so as to minimize the sum of their lengths?

Solution. Letting x and y be the length of the sides of the right triangle, the sum of the length of these legs is $x + y$ and the area is $(1/2)xy$. Thus we want to minimize $x + y$, subject to the condition $(1/2)xy = 100$. That is the same as saying $y = 200/x$; we seek the minimum of $L = x + 200/x$. By problem 6 (taking $C = 200$, the minimum value occurs when $x = \sqrt{200} = 10\sqrt{2}$, and the minimum length is

$$L_{min} = 10\sqrt{2} + \frac{200}{10\sqrt{2}} = 20\sqrt{2} .$$