The Equivalence of Two Forms of the Canonical Element Conjecture

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We wish to show that the following two conjectures are equivalent:

A. (M. Hochster). Let (R,m,k) be a local ring, where d is the Krull dimension of R, and let

$$(*) \qquad 0 \longrightarrow S \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k \longrightarrow 0$$

be exact, with F_i free for $0 \leq i \leq d-1$, so that S is a $d^{\underline{th}}$ module of syzygies of k. The sequence (*) defines an element of $\mathrm{Ext}^d_R(k,S)$, which maps to an element η in the local cohomology $H^d_{\underline{m}}(S)$. Conjecture: $\eta \neq 0$.

B. Let $x_1,\ldots,x_d$ be a system of parameters for R. For all $n \geq 1$, let $K^n_\bullet$ denote the Koszul complex on $x^n_1,\ldots,x^n_d$, and let $G^n_\bullet$ be a free resolution of $R/(x^n_1,\ldots,x^n_d)$. The identity map on $R/(x^n_1,\ldots,x^n_d)$ induces a map of complexes from $K^n_\bullet$ to $G^n_\bullet$, which we denote ϕ. Then $K^n_d \cong R$, and $\phi_d(1)$ defines an element ξ in $\mathrm{Tor}^R_d(k,R/(x^n_1,\ldots,x^n_d))$ which is unique up to multiplication by a unit. Conjecture: $\xi \neq 0$.

The relation between these two conjectures comes through the description of local cohomology using the complex $C_\bullet$ defined as follows:

$$C_{d-k} = \bigoplus_{1\le i_1<\ldots<i_k\le d} A_{x_{i_1},\ldots,x_{r_k}}$$

$d_{d-k}\colon C_{d-k} \longrightarrow C_{d-k-1}$ on each component

$A_{x_{i_1}\ldots x_{i_k}} \longrightarrow A_{x_{j_1}\ldots x_{j_{k+1}}}$ is $(-1)^r$ times the map induced by localization if $\{i_1\ldots i_t\} = \{j_1\ldots \hat{j}_r,\ldots,j_{k+1}\}$; and 0 otherwise.

The connection with conjecture A comes from the isomorphism: $H^k_m(M) \cong H_{d-k}(C.\otimes M)$ for all modules M.

The connection with conjecture B comes from the isomorphism: $C. \cong \varinjlim_n K^n_{\bullet}$ where the map: $K^n_{\bullet} \longrightarrow K^{n+1}_{\bullet}$ takes the element $e_{i_1}\wedge\ldots\wedge e_{i_k}$ to $x_{j_1}x_{j_2}\ldots x_{j_{d-k}}\, e_{i_1}\wedge\ldots\wedge e_{i_k}$ where $\{j\text{'s}\}$ is the complement of $\{i\text{'s}\}$ in $\{1,2,\ldots,d\}$.

Let F denote the complex $0 \longrightarrow F_{d-1} \longrightarrow\ldots\longrightarrow F_0 \longrightarrow k(=F_{-1}) \longrightarrow 0$. Then the homology of $F.$ is S in degree $d-1$, and we have

$$H^d_m(S) \cong H_{d-1}(C.\otimes F.).$$

The first step in showing the equivalence of Conjectures A and B is the identification of η with an element in $H_{d-1}(C.\otimes F.)$. First, the sequence (*) can be mapped into an injective revolution

$$0 \longrightarrow S \longrightarrow I_{d-1} \longrightarrow I_{d-2} \longrightarrow\ldots\longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow\ldots$$

of S (strangely numbered to correspond to $F.$), and if the map is

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & S & \longrightarrow & F_{d-1} & \longrightarrow & F_{d-2} & \longrightarrow \cdots \longrightarrow & F_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \psi & & \\
0 & \longrightarrow & S & \longrightarrow & I_{d-1} & \longrightarrow & I_{d-2} & \longrightarrow \cdots \longrightarrow & I_0 & \longrightarrow & I_{-1} & \longrightarrow & I_{-2} \longrightarrow \cdots
\end{array}$$

then the element of $H_{-1}(\mathrm{Hom}(k,I.))$ corresponding to the extension (*) in $\mathrm{Ext}^d(k,S)$ is the class of the map ψ. (See for instance MacLane Homology Theorem). Note that ψ can be identified with an element x in I_{-1} annihilated by m; it is zero in $\mathrm{Ext}^d(k,S)$ if and only if x can be lifted to an element of I_0 annihilated by m, and η is zero in $H^d_m(S)$ if and only if x can be lifted to an element of I_0 annihilated by some power of η.

Let $I. = 0 \longrightarrow I_{d-1} \longrightarrow I_{d-2} \longrightarrow \cdots$. Then $H^d_m(S) \cong H_{d-1}(C.\otimes I.)$ Furthermore, $1\otimes x \in C_d \otimes I_{-1}$ is a cycle in $(C.\otimes I.)_{d-1}$, and we claim that its class in homology is η. To see this, consider the spectral sequence obtained from $C. \otimes I_k$ for each k. This degenerates, leaving

$$\begin{array}{ccccccc}
0 \longrightarrow & \Gamma_m(I_{d-1}) & \longrightarrow \cdots & \Gamma_m(I_0) & \longrightarrow & \Gamma_m(I_{-1}) & \longrightarrow \cdots \\
 & \| & & & & \|\wr & \\
0 \longrightarrow & \Gamma_m(I_{d-1})\otimes C_d & \longrightarrow & \cdots & \longrightarrow & \Gamma_m(I_{-1}\otimes C_d) & \longrightarrow \cdots
\end{array}$$

where Γ_m denotes elements annihilated by a power of m. Thus $1 \otimes x$ corresponds to the element $x \in I_{-1}$, and from the above discussion the class of x is η.

We now return to $F.$. We have a map $F. \longrightarrow I.$ which sends $\overline{1} \in F_{-1} \cong k$ to $x \in I_{-1}$. Since $F. \longrightarrow I.$ induces an isomorphism in homology and $C.$ is a complex of flat modules, we have isomorphisms

$$H_*(C_\bullet \otimes F_\bullet) \longrightarrow H_*(C_\bullet \otimes I_\bullet).$$

Furthermore, the cycle $1\otimes\overline{1}$ goes to $1\otimes x$, so we can identify η with the class of $1\otimes\overline{1}$ in $H_{d-1}(C_\bullet \otimes F_\bullet)$.

We now use the fact that $C_\bullet \cong \varinjlim_n K^n_\bullet$. Note that in degree d, we have
$$\begin{array}{ccc} K^n_\bullet & \longrightarrow & K^{n+1}_\bullet \\ \| & & \| \\ R & \xrightarrow{1} & R \end{array}$$
, and $C_d = \varinjlim (R \xrightarrow{1} R \xrightarrow{1} \ldots) = R$. Hence the element $1\otimes\overline{1}$ in $C_\bullet \otimes F_\bullet$ can be lifted to $1\otimes\overline{1} \in K^n_\bullet \otimes F_\bullet$ for any n. Also, but the commutativity of $\varinjlim$ with $\otimes$ and homology, we have

$$H_{d-1}(C_\bullet \otimes F_\bullet) \cong \varinjlim_n H_{d-1}(K^n_\bullet \otimes F_\bullet).$$

Hence conjecture A can be reformulated: for every $n \geq 1$, the class of $1\otimes\overline{1}$ in $H_{d-1}(K^n_\bullet \otimes F_\bullet)$ is not zero.

Now let $G^n_\bullet$ be a free resolution of $R/(x_1^n,\ldots,x_d^n)$ as in Conjecture B. The map $\phi_\bullet : K^n_\bullet \longrightarrow G^n_\bullet$ induces a map $\phi_\bullet\otimes 1$: $K^n_\bullet \otimes F_\bullet \longrightarrow G^n_\bullet \otimes F_\bullet$. Furthermore, the quasi-isomorphism: $S \longrightarrow F_\bullet$ induces quasi-isomorphism,

$$K^n_\bullet \otimes S \longrightarrow K^n_\bullet \otimes F_\bullet \quad \text{and} \quad G^n_\bullet \otimes S \longrightarrow G^n_\bullet \otimes F_\bullet,$$

and we have a commutative diagram:

$$\begin{array}{ccc} H_{d-1}(K^n_\bullet \otimes F_\bullet) & \longrightarrow & H_{d-1}(G^n_\bullet \otimes F_\bullet) \\ \uparrow\wr & & \uparrow\wr \\ H_{d-1}(K^n_\bullet \otimes S) & \longrightarrow & H_{d-1}(G^n_\bullet \otimes S). \end{array}$$

But $H_{d-1}(K_\bullet^n \otimes S) \cong R/_{(x_1^n,\ldots,x_d^n)} \otimes S \cong H_{d-1}(G_\bullet^n \otimes S)$, so we can conclude that the map induced in H_{d-1} by $\phi_\bullet \otimes 1$ is an isomorphism. Thus $\eta \neq 0$ if and only if the image of $1\otimes\overline{1}$ in $H_{d-1}(G_\bullet^n \otimes F_\bullet)$ is not zero.

We now examine the spectral sequence of $G_\bullet^n \otimes F_\bullet$. The double complex looks like: (writing G_i for G_i^n):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G_0 \otimes F_{d-1} & \longrightarrow \cdots \longrightarrow & G_0 \otimes F_0 & \longrightarrow & G_0 \otimes k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \vdots\uparrow & & \\
 & & \vdots & & \vdots & & G_{d-1} \otimes k & & \\
 & & & & & & \uparrow & & \\
0 & — & G_d \otimes F_{d-1} & — \cdots — & G_d \otimes F_0 & — & G_d \otimes k & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & G_{d+1} \otimes k & & \\
 & & & & & & \uparrow & &
\end{array}
$$

Now the image of $1\otimes\overline{1}$ in $G_d\otimes k$ is just $\phi_d(1)\otimes\overline{1}$. If we now take the homology of the columns in this diagram, the class of $\phi_d(1)\otimes\overline{1}$ will be ξ in the diagram:

$$
\begin{array}{llllll}
0 \longrightarrow R/_{(x_i^n)} \otimes F_{d-1} & \longrightarrow \cdots \longrightarrow & R/_{(x_i^n)} \otimes F_0 & \longrightarrow & \mathrm{Tor}_0(R/_{(x_i^n)},k) & \longrightarrow 0 \\
0 \;.\;.\;. \quad 0 \;.\;.\;. & & 0 & \longrightarrow & \mathrm{Tor}_1(R/_{(x_i^n)},k) & \longrightarrow \\
\qquad\quad \vdots & & & & \quad\vdots & \\
\qquad\quad 0 \;.\;.\;. & & 0 & & \xi \in \mathrm{Tor}_d(R/_{(x_i^n)},k) & \longrightarrow \\
 & & & & \quad\vdots &
\end{array}
$$

All maps to or from $\text{Tor}_d(R/_{(x_i^n)},k)$ in future stages of the spectral sequence are zero. Hence $\xi \neq 0 \Leftrightarrow \xi$ survives forever in the spectral sequence $\Leftrightarrow \xi$ defines a non-zero element in $H_{d-1}(G_{\bullet}^n \otimes F_{\bullet}) \Leftrightarrow \eta \neq 0$. Thus Conjectures A and B are equivalent.