INTRODUCTION TO ENDOSCOPY

Snowbird Lectures
Utah, June 2006

Jean-Pierre Labesse
Institut Mathématique de Luminy, UMR 6206
Marseille, France

Foreword

Many questions about non commutative Lie groups boil down to questions in invariant harmonic analysis, the study of distributions on the group that are invariant by conjugacy. The fundamental objects of invariant harmonic analysis are orbital integrals and characters, respectively the geometric and spectral sides of the trace formula.

In the Langlands program a cruder form of conjugacy called **stable conjugacy** plays a role. The study of Langlands functoriality often leads to correspondences that are defined only up to stable conjugacy. Endoscopy is the name given to a series of techniques aimed to investigate the difference between ordinary and stable conjugacy.

Let G be a reductive group over a field F. Recall that one says that γ and γ' in $G(F)$ are conjugate if there exists $x \in G(F)$ such that $\gamma' = x\gamma x^{-1}$. Roughly speaking, stable conjugacy amounts to conjugacy over the algebraic closure $\overline{F}$: at least for strongly regular semisimple elements, one says that γ and γ' in $G(F)$ are stably conjugate if there is $x \in G(\overline{F})$ such that $\gamma' = x\gamma x^{-1}$.

On the geometric side, the basic objects of stably invariant harmonic analysis are stable orbital integrals. On the spectral side, the notion of L-packets of representations is the stable analogue for characters of tempered representations. The case of non tempered representations is the subject of conjectures of Arthur that will be examined in Vogan's lectures.

The word "endoscopy" has been coined to express that we want to see ordinary conjugacy **inside** stable conjugacy. We shall introduce the basic notions of local endoscopy: κ-orbital integrals, endoscopic groups, endoscopic transfer of orbital integrals and its dual for characters with an emphasis on the case of real groups, following the work of Diana Shelstad.

I. $GL(2,\mathbb{R})$ VERSUS $SL(2,\mathbb{R})$

I.1 – Representations of $GL(2,\mathbb{R})$

By representation we understand admissible representations. Their classification is as follows: all admissible irreducible representations of $GL(2,\mathbb{R})$ are subquotients of principal series

$$\rho(\mu_1,\mu_2)$$

where μ_i are characters of $\mathbb{R}^\times$. They are induced from the Borel subgroup i.e. it is the right regular representation in the space of smooth functions such that

$$f(\begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix} g) = \mu_1(\alpha)\mu_2(\beta)\left|\frac{\alpha}{\beta}\right|^{1/2} f(g)$$

We have three types of subquotients according to the value of

$$\mu = \mu_1\mu_2^{-1}$$

We shall restrict ourselves to representations that induce unitary characters on the center and hence we assume that the product $\mu_1\mu_2$ is unitary.

1 – Irreducible principal series

$$\pi(\mu_1,\mu_2)$$

where

$$\mu \neq x^n \,.\, \mathrm{sign}\,(x)$$

for some $n \in \mathbb{Z} - \{0\}$. These representations are unitarizable if μ is unitary or if $\mu = |x|^s$ with s real and $-1 < s < 1$.

2 – Finite dimensional subquotients

$$\pi(\mu_1,\mu_2)$$

when $\mu = x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z} - \{0\}$. It is unitarizable if $n = \pm 1$.

3 – Discrete series subquotients

$$\sigma(\mu_1,\mu_2)$$

when $\mu = x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z} - \{0\}$. These representations are unitarizable.

The various representations are equivalent under permutation of the μ_i: $\pi(\mu_1,\mu_2) \simeq \pi(\mu_1,\mu_2)$ etc...

I.2 – Langlands parameters for $GL(2,\mathbb{R})$

We first have to introduce the Weil group for $\mathbb{R}$. This is the subgroup matrices in $SU(2)$ generated by

$$\left\{\mathbf{z} = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\ ,\ z \in \mathbb{C}^\times\right\} \quad \text{and} \quad w_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The element w_σ acts by conjugacy as the non trivial element in the Galois group, in other words there is an exact sequence

$$1 \to \mathbb{C}^\times \to \mathbb{W}_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

and a section of the map $\mathbb{W}_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is defined by

$$\sigma \mapsto w_\sigma$$

Observe that $w_\sigma^2 = -\mathbf{1}$ and hence the above extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is the non trivial one.

A Langlands parameter for $GL(2,\mathbb{R})$ is a conjugacy classes of homomorphisms of $\mathbb{W}_\mathbb{R}$ in $GL(2,\mathbb{C})$ with semisimple images.

For $z = \rho e^{i\theta}$ let $\chi_{s,n}(z) = \rho^s e^{in\theta}$ then, up to conjugacy, the admissible maps φ are of the following form:

1 – for some $s_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}_2$

$$\varphi_{s_1,m_1,s_2,m_2}(\mathbf{z}) = \begin{pmatrix} \chi_{s_1,0}(z) & 0 \\ 0 & \chi_{s_2,0}(z) \end{pmatrix}$$

with

$$\varphi_{s_1,m_1,s_2,m_2}(w_\sigma) = \begin{pmatrix} (-1)^{m_1} & 0 \\ 0 & (-1)^{m_2} \end{pmatrix}$$

Up to conjugacy $\varphi_{s_1,m_1,s_2,m_2} \simeq \varphi_{s_2,m_2,s_1,m_1}$.

2 – for some $s \in \mathbb{C}$ and $n \in \mathbb{Z}$

$$\varphi_{s,n}(\mathbf{z}) = \begin{pmatrix} \chi_{s,n}(z) & 0 \\ 0 & \chi_{s,-n}(z) \end{pmatrix}$$

with

$$\varphi_{s,n}(w_\sigma) = \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}$$

Up to conjugacy $\varphi_{s,n} \simeq \varphi_{s,-n}$.

The intersection of the two sets of conjugacy classes of maps is the class of parameters of the form

$$\varphi_{s,0} \simeq \varphi_{s,1,s,0} \simeq \varphi_{s,0,s,1}$$

Let us denote by ε the homomorphism from $\mathbb{W}_{\mathbb{R}}$ to $\mathbb{C}^\times$ defined by

$$\varepsilon(\mathbf{z}) = 1 \quad \text{and} \quad \varepsilon(w_\sigma) = -1$$

Lemma I.1 – *If φ is a Langlands parameter, then*

$$\varphi \otimes \varepsilon \simeq \varphi$$

if and only if φ is in the class of $\varphi_{s,n}$ for some s and some n.

Proof: First it is clear that

$$\varphi_{s,n} \otimes \varepsilon = \alpha\, \varphi_{s,n}\, \alpha^{-1} \quad \text{where} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conversely we have to observe that

$$\varphi_{s_1,m_1,s_2,m_2} \not\simeq \varphi_{s_1,m_1+1,s_2,m_2+1}$$

unless $s_1 = s_2$ and $m_1 + m_2 \equiv 1$

$\square$

The correspondence between irreducible representations and Langlands parameters for $GL(2,\mathbb{R})$ is obtained as follows. Recall the correspondence for $GL(1,\mathbb{R})$ (also called Tate Nakayama duality): a character μ of $GL(1,\mathbb{R}) = \mathbb{R}^\times$ of the form

$$\mu(x) = |x|^s \operatorname{sign}(x)^m$$

correspond to a character of the Weil group

$$\mathbf{z} \mapsto \chi_{s,0} \quad \text{and} \quad w_\sigma \mapsto (-1)^m$$

We get a natural bijection between equivalence classes of admissible irreducible representations of $GL(2,\mathbb{R})$ and conjugacy classes of admissible homomorphisms of $\mathbb{W}_{\mathbb{R}}$ in $GL(2,\mathbb{C})$ as follows:

$$\pi(\mu_1, \mu_2) \mapsto \varphi_{s_1,m_1,s_2,m_2}$$

with

$$\mu_i(x) = |x|^{s_i} \operatorname{sign}(x)^{m_i}$$

and

$$\sigma(\mu_1, \mu_2) \mapsto \varphi_{s,n}$$

with

$$\mu_1\mu_2(x) = |x|^{2s} \operatorname{sign}(x)^{n+1}$$

and

$$\mu_1\mu_2^{-1}(x) = x^n \operatorname{sign}(x)$$

A parameter corresponds to a tempered representation if the image of the map is bounded i.e. if the s_i are imaginary.

I.3 – Representations of $SL(2\mathbb{R})$

The representations of $SL(2,\mathbb{R})$ and their L-packets are easily understood in terms of those for $GL(2,\mathbb{R})$. In fact it is an easy exercise to show that any irreducible representation of $SL(2,\mathbb{R})$ occurs in the restriction of an irreducible representation of $GL(2,\mathbb{R})$. Those restrictions either remain irreducible (which is the case for principal series for generic values of the parameter) or split into two irreducible components whose union is an L-packet for $SL(2,\mathbb{R})$.

Two representations π and π' are in the same L-packet if and only if, up to equivalence, they are conjugated by α:

$$\pi' \simeq \pi \circ \mathrm{Ad}(\alpha) \quad \text{where} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have the following classification:

1 – Irreducible principal series: they are the $\pi(\mu)$ obtained by restriction of $\pi(\mu_1,\mu_2)$ with $\mu \neq x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z}$.

2 – Finite dimensional representations of dimension n: they are the $\pi(\mu)$ obtained by restriction of $\pi(\mu_1,\mu_2)$ with $\mu = x^n \,.\, \mathrm{sign}\,(x)$ and $n \neq 0$.

3 – Discrete series L-packets

$$\sigma(\mu) = \{D^+_{|n|}, D^-_{|n|}\}$$

are obtained by restriction of $\sigma(\mu_1,\mu_2)$ with

$$\mu = x^n \,.\, \mathrm{sign}\,(x)$$

and $n \neq 0$.

4 – Limits of discrete series L-packet

$$\sigma(\mu) = \{D^+_0, D^-_0\}$$

is obtained by restriction of $\pi(\mu_1,\mu_2)$ with

$$\mu = \mathrm{sign}\,(x)$$

The L-packets of representations that are indexed by characters μ and μ^{-1} are equivalent.

The minimal K-type for $D^{\pm}_n$ is $\pm(n+1)$ i.e.

$$r(\theta) \mapsto \exp\left(\pm i\,(n+1)\,\theta\right)$$

if

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The character of D_n^+ on $K = SO(2,\mathbb{R})$ is given by

$$\Theta_n^+(r(\theta)) = \frac{e^{i(n+1)\theta}}{1 - e^{2i\theta}} = \frac{-e^{in\theta}}{e^{i\theta} - e^{-i\theta}}$$

while the character of D_n^- is the complex conjugate:

$$\Theta_n^-(r(\theta)) = \frac{e^{-i(n+1)\theta}}{1 - e^{-2i\theta}} = \frac{e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

I.4 – Langlands parameters for $SL(2,\mathbb{R})$

From the bijection between equivalence classes of representations and conjugacy classes of Langlands parameters for $GL(2,\mathbb{R})$ one derives a bijection between equivalence classes of L-packets of admissible irreducible representations of $SL(2,\mathbb{R})$ and conjugacy classes of admissible homomorphisms of $\mathbb{W}_{\mathbb{R}}$ in $PGL(2,\mathbb{C})$. We need the

Lemma I.2 – *Any projective representation of* $\mathbb{W}_{\mathbb{R}}$ *lifts to a representation.*

Proof: This is a particular case of a result of [Lab1]. This particular case can also be found in [Lan2]. We shall prove it only for two dimensional representations. Consider a two-dimensional projective parameter. The image of $\mathbb{C}^\times$ is inside a torus. Now this image is either trivial or is a maximal torus in $PGL(2,\mathbb{C})$ and hence of dimension 1. If this image is trivial the lemma is easy to prove. If this image is a torus the map restricted to $\mathbb{C}^\times$ is given by a non trivial character $\chi_{s,n}$. If $n=0$ th existence of a lift is easy. If $n\neq 0$ then and the image of w_σ must lie in the normalizer of the torus and acts non trivially and hence its square is central i.e. projectively trivial and $s=0$. Since $w_\sigma^2=-1$ this shows that n is even and hence $\varphi_{0,n/2}$ is a lift.

□

The correspondence is a follows:

1 – the parameter for $\pi(\mu)$ is the conjugacy class of the projective parameter $\varphi_{s,m}$ defined by $\varphi_{s,m,0,0}$ with

$$\mu(x)=|x|^s\,\mathrm{sign}\,(x)^m$$

2 – the parameter for $D_n^\pm$ is the conjugacy class of the projective parameter φ_n defined by $\varphi_{0,n}$.

We have seen in lemma I.1 that

$$\varphi_{0,n}\otimes\varepsilon=\alpha\varphi_{0,n}\alpha^{-1}\quad\text{where}\quad\alpha=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

But ε has a central image and hence the projective parameters defined by $\varphi_{0,n}$ and $\varphi_{0,n}\otimes\varepsilon$ are equal. This shows that the projective image of α belongs to the centralizer of the projective image of $\varphi_{0,n}$.

Let φ_n be the projective parameter defined by $\varphi_{0,n}$ and let S_{φ_n} denote the centralizer of the image of φ_n and $\mathfrak{S}_{\varphi_n}$ the quotient of S_{φ_n} by its connected component $S^0_{\varphi_n}$ times the center $Z_{\check{G}}$ of $\check{G}$ (trivial here). When $n\neq 0$ we have

$$\mathfrak{S}_{\varphi_n}=S_{\varphi_n}\simeq\{1,\alpha\}$$

When $n=0$ the group $S^0_{\varphi_0}$ is a torus but again $\mathfrak{S}_{\varphi_0}$ is generated by the image of α.

II. Endoscopy: the simplest case

II.1 – Endoscopy for $SL(2,\mathbb{R})$

We shall first discuss endoscopy for $SL(2)$. This is the simplest instance of endoscopy and in fact it is in this case that endoscopy was first observed. As far as I know (but Bill Casselman may know more), endoscopy was discovered by Langlands while he was trying to understand the Zeta function of some simple Shimura varieties: those attached to compact inner forms of groups between $GL(2)$ and $SL(2)$ over a totally real field.

Endoscopy arises because the conjugacy over F and over $\overline{F}$ (the algebraic closure of F) may be different. For example inside $SL(2,\mathbb{R})$ the two rotations

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$r(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

are not conjugate (unless $\theta \in \mathbb{Z}\,\pi$) although they are conjugate inside $SL(2,\mathbb{C})$ and also inside $GL(2,\mathbb{R})$ by

$$w = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The union of the sets of conjugates of elements in the pair $(r(\theta), r(-\theta))$ is said to be a **stable conjugacy class**. Similarly the unipotent elements

$$u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u_0^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

are conjugate in $SL(2,\mathbb{C})$ but not in $SL(2,\mathbb{R})$. There are two conjugacy classes and only one stable conjugacy class of (non trivial) unipotent elements.

Endoscopy on the spectral side is also easily observed for $SL(2,\mathbb{R})$: discrete series and limits of discrete series representations come by pairs called L-packets.

In all above examples the objects inside pairs are exchanged by conjugacy under $w = i\alpha$ an element in the normalizer of $SO(2)$ in $SL(2,\mathbb{C})$. We observe that if σ is the non trivial element of the Galois group

$$a_\sigma = w\sigma(w)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates a subgroup of order 2 that can be identified with $H^1(\mathbb{C}/\mathbb{R}, SO(2))$. The characters of this 2-group are called endoscopic character. The endoscopic groups for $SL(2,\mathbb{R})$ correspond to these two characters. The "trivial" endoscopic group is $SL(2,\mathbb{R})$ himself while the non trivial one is $T(\mathbb{R}) = SO(2,\mathbb{R})$ the compact torus.

Finally we observe that the sum

$$S\Theta_n(r(\theta)) = \Theta_n^+(r(\theta)) + \Theta_n^-(r(\theta)) = -\frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

is invariant under conjugacy by w, and we say that $S\Theta_n$ is a stable character, while

$$\Delta(r(\theta))(\Theta_n^+ - \Theta_n^-)(r(\theta)) = e^{in\theta} + e^{-in\theta}$$

is the sum of two characters of $T(\mathbb{R})$ where

$$\Delta(r(\theta)) = -2i \sin\theta$$

For further generalization we observe that

$$\Delta(r(\theta)) = -e^{i\theta}(1 - e^{-2i\theta}) = -i \ \mathrm{sign}\,(\sin(\theta))|e^{i\theta} - e^{-i\theta}|$$

The group $G^* = SL(2)$ over $\mathbb{R}$ has an inner form G such that $G(\mathbb{R}) = SU(2)$. Observe that $SL(2,\mathbb{R})$ and $SU(2)$ have in common the compact Cartan subgroup

$$T(\mathbb{R}) = SO(2)$$

The conjugacy classes in $SU(2)$ are classified by the pairs of eigenvalues $\{e^{i\theta}, e^{-i\theta}\}$. Hence they are in bijection with elliptic stable conjugacy classes in $SL(2,\mathbb{R})$. This correspondence is dual to the correspondence between representation F_n of dimension n of $SU(2)$ and L-packets of discrete series $D_n^\pm$ and there is the character identity

$$\mathrm{trace}\ F_n(r(\theta)) = -S\Theta_n(r(\theta))$$

One should observe that inside $GL(n,F)$ conjugacy over F and over $\overline{F}$ coincide. Hence there is no non trivial endoscopy for $GL(n)$ in the sense that conjugacy and stable conjugacy coincide and L-packets are singletons. Nevertheless, if we consider now an inner forms G of $G^* = GL(n)$, then there is a correspondence for conjugacy classes and for representations between G and G^*. These correspondences, often called Jacquet-Langlands correspondences, are nowadays considered as an instance of endoscopy.

II.2 – Asymptotic behaviour of orbital integrals

Let f be a smooth and compactly supported function on $G = SL(2,\mathbb{R})$. We are to study its orbital integrals. For $\gamma \in G$ let us denote by G_γ its centralizer in G. The orbital integral is

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, d\dot{x}$$

Observe it depends on the choice of Haar measures on G and G_γ. We shall study the asymptotic behaviour of $\mathcal{O}_\gamma(f)$ when γ is diagonal and $\gamma \to 1$ and when $\gamma = r(\theta)$ and $\theta \to 0$. We may and will assume that f is K-central i.e. $f(kx) = f(xk)$.

Consider the first case where γ is diagonal:

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $ab = 1$. Then

$$\mathcal{O}_\gamma(f) = \int_U f(u^{-1}\gamma u)\, du$$

for a standard choice of Haar measures, where U is the group of unipotent matrices and hence

$$\mathcal{O}_\gamma(f) = \int_{\mathbb{R}} f\begin{pmatrix} a & (a-b)x \\ 0 & b \end{pmatrix} dx$$

which yields the

Lemma II.1 – *Let* $\Delta(\gamma) = |a-b|$ *then*

$$h(\gamma) = \Delta(\gamma)\mathcal{O}_\gamma(f)$$

extends to a smooth function on the group $A(\mathbb{R})$ *of diagonal real matrices.*

This is the simplest example of endoscopic geometric transfer for a non elliptic endoscopic group.

When $\gamma = r(\theta)$ the orbital integral can be computed using the Cartan decomposition $G = KAK$. Here we may take $K = T(\mathbb{R})$ the group of rotations. We have

$$\mathcal{O}_{r(\theta)}(f) = c\, F(\sin\theta)$$

with c a constant that depends on the choice of Haar measures and

$$F(\lambda) = \int_1^\infty f\begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \left|t - t^{-1}\right| \frac{dt}{t}$$

where

$$a(\lambda) = \sqrt{1-\lambda^2}$$

Observe that since f is K-central we have for any a, b and c

$$f\begin{pmatrix} a & b \\ c & a \end{pmatrix} = f\begin{pmatrix} a & -c \\ -b & a \end{pmatrix}$$

and hence

$$F(\lambda) = \int_0^\infty \varepsilon(t-1) f\begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} dt$$

with $\varepsilon(x) = \mathrm{sign}(x)$. To study the asymptotic its behaviour when $\lambda \to 0$ we consider

$$A(\lambda) = \int_0^\infty \varepsilon(t-1) f\begin{pmatrix} a(\lambda) & \lambda t \\ 0 & a(\lambda) \end{pmatrix} dt$$

By Taylor-Lagrange formula

$$F(\lambda) = A(\lambda) + \lambda B(\lambda)$$

where

$$B(\lambda) = \int_0^\infty \varepsilon(t-1) g\begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \frac{dt}{t}$$

for some smooth function g compactly supported in the upper right variable and that has a $O(u)^{-1}$ decay in the lower left variable u so that the integral is absolutely convergent.

Observe that

$$A(\lambda) = |\lambda|^{-1} \int_0^\infty f\begin{pmatrix} 1 & \varepsilon(\lambda)u \\ 0 & 1 \end{pmatrix} du - 2f(\mathbf{1}) + o(\lambda)$$

Since $B(\lambda)$ is at most of logarithmic growth we see that the even functions

$$G(\lambda) = |\lambda|(F(\lambda) + F(-\lambda))$$

and

$$H(\lambda) = \lambda(F(\lambda) - F(-\lambda))$$

extend to continuous functions at the $\lambda = 0$.

We need more precise informations on the asymptotic behaviour. Hence we have to look more carefully to the error term $B(\lambda)$ which, we recall, equals:

$$\int_0^\infty \varepsilon(t-1) g\begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \frac{dt}{t}$$

But this is the difference of two terms whose leading terms are equivalent to $\ln(|\lambda|^{-1})g(\mathbf{1})$ up to continuous terms. Hence B is continuous. Generalizing this process one gets asymptotic expansions of the following form:

$$G(\lambda) = \sum_{n=0}^{N} \left(a_n\,|\lambda|^{-1} + b_n\right)\lambda^{2n} + o(\lambda^{2N})$$

and

$$H(\lambda) = \sum_{n=0}^{N} h_n \, \lambda^{2n} + o(\lambda^{2N})$$

and hence $H(\lambda)$ is smooth. We have proved the following

Lemma II.2 – *There is a smooth fonction h on $T(\mathbb{R})$ such that:*

$$h(\gamma) = \Delta(\gamma) \left(O_\gamma(f) - O_{w(\gamma)}(f) \right)$$

for $\gamma = r(\theta) \in T(\mathbb{R})$ and

$$\Delta(r(\theta)) = -2i \sin\theta$$

This lemma establishes the simplest case of a non trivial geometric endoscopic transfer for an elliptic endoscopic group. A variant of this lemma establishes the transfer between orbital integrals weighted by the character of order two

$$\omega = \mathrm{sign} \circ \det$$

for functions on $GL(2,\mathbb{R})$:

$$\mathcal{O}_\gamma^\omega(f) = \int_{G_\gamma \backslash G} \omega(x) f(x^{-1}\gamma x) \, d\dot{x}$$

and functions on the elliptic maximal torus

$$\widetilde{T}(\mathbb{R}) \simeq \mathbb{C}^\times = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2,\mathbb{R}) \right\}$$

Lemma II.3 – *There is a smooth fonction h on $\widetilde{T}(\mathbb{R})$ such that:*

$$h(\gamma) = \Delta(\gamma) O_\gamma^\omega(f)$$

for $\gamma = \rho \, r(\theta) \in \widetilde{T}(\mathbb{R})$ and

$$\Delta(r(\theta)) = -2i \sin\theta$$

An other simple example is given by the transfer between inner forms.

Consider the compact group $G = SU(2,\mathbb{R})$. This is an inner form of $G^* = SL(2,\mathbb{R})$. In fact one has $g \in SU(2,\mathbb{R})$ iff

$$g \in SL(2,\mathbb{C}) \quad \text{and} \quad g = J\sigma(g)J^{-1}$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\sigma(g)$ is the complex conjugate of g.

There is an injection of the set of conjugacy classes inside G into the set of stable conjugacy classes in G^* induced by the following correspondence: given $\gamma \in SU(2,\mathbb{R})$ we denote by $\tilde\gamma$ any semisimple element $\tilde\gamma \in SL(2,\mathbb{R})$ with the same eigenvalues.

Lemma II.4 – *Given $f \in \mathcal{C}_c^\infty(SU(2,\mathbb{R}))$ there is a function $\tilde f \in \mathcal{C}_c^\infty(SL(2,\mathbb{R}))$ such that for $\gamma \in SU(2,\mathbb{R})$ non central*

$$SO_{\tilde\gamma}(\tilde f) = -O_\gamma(f)$$

where

$$SO_\gamma(\tilde f) = \sum SO_{w(\tilde\gamma)}(\tilde f)$$

and the sum is over the complex Weyl group for $SO(2) \subset SL(2)$ and

$$SO_{\tilde\gamma}(\tilde f) = O_{\tilde\gamma}(\tilde f) = 0$$

if $\tilde\gamma \in SL(2,\mathbb{R})$ has distinct real eigenvalues.

An easy proof is obtained using pseudo-coefficients. This will be explained in a fairly general case in a section toward the end of these notes.

In the case of $GL(2)$ a similar result is used for all local fields in Jacquet-Langlands. This lemma is a particular case of the general result due to Shelstad [She1] one the transfer between inner forms of real groups.

III. The general case

III.1 – Group cohomology and hypercohomology

Consider first a set X with an action of a group Γ. By definition $H^0(\Gamma, X)$ is the fixed point set X^Γ: the set of $x \in X$ such that

$$x = \sigma(x) \quad \text{for all} \quad \sigma \in \Gamma$$

Now let G be an algebraic group with an action of Γ compatible with the group structure. Then $H^0(\Gamma, G)$ is a group. Moreover we may define a set $H^1(\Gamma, G)$ as the quotient of the set $Z^1(\Gamma, G)$ of 1-cocycles i.e. maps

$$\sigma \in \Gamma \mapsto a_\sigma \in G$$

such that

$$a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = 1$$

by the equivalence relation

$$a'_\sigma \simeq a_\sigma \Longleftrightarrow a'_\sigma = b\, a_\sigma\, \sigma(b)^{-1}$$

for some $b \in G$. The set $H^1(\Gamma, G)$ has no natural group structure. But the trivial class makes it a pointed set. Finally, if A is an abelian group, we have the familiar cohomology groups in all degrees

$$H^i(\Gamma, A)$$

This can be slightly generalized as follows. One can define a hyper-cohomological theory in degrees ≤ 0 for complexes of the form

$$[G \Rightarrow X]$$

where G is a group acting (on the left) on a set X with a group Γ acting on the complex. The cohomology set

$$H^0(\Gamma, G \Rightarrow X)$$

it the quotient of the set of pairs (x, a) with $x \in X$ and a a map from Γ to X with

$$x = a_\sigma \,.\, \sigma(x) \quad \text{and} \quad a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = 1$$

by the equivalence relation

$$(x', a') \simeq (x, a) \Longleftrightarrow x' = bx \quad \text{and} \quad a'_\sigma = b\, a_\sigma\, \sigma(b)^{-1}$$

There is also an hyper-cohomological theory in degrees ≤ 1 for complexes of groups of the form

$$[G' \to G]$$

that are "left crossed modules". This means that G and G' are groups and we are given not only a homomorphism $G' \to G$, which yields an action of G' on G by left translations, but also an action of G on G' and that those two actions are compatible with adjoint actions. This implies in particular that

$$\ker[G' \to G]$$

is an abelian group.

A morphism of crossed modules

$$[G' \to G] \to [H' \to H]$$

is a pair of maps $G' \to H'$ and $G \to H$ that intertwines the various actions, in particular we have a commutative diagram

$$\begin{array}{ccc} G' & \to & G \\ \downarrow & & \downarrow \\ H' & \to & H \end{array}$$

Such a morphism is a quasi-isomorphism if it induces isomorphisms for kernels and cokernels.

If we have an action of a group Γ, compatible with the structure of crossed module, one has a cohomology theory

$$H^i(\Gamma, G' \to G)$$

in degrees $i \leq 1$. The H^1 is defined as a quotient of the set of 1-hyper-cocycles (a, b) with

$$a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = \rho(b_{\sigma,\tau})$$

where ρ is the homomorphism $G' \to G$ and

$$a_\sigma * \sigma(b_{\tau,\mu}) \,.\, b_{\sigma,\tau\mu} = b_{\sigma,\tau} \,.\, b_{\sigma\tau,\mu}$$

where $*$ denotes the action of G on G'. The equivalence relation is

$$a'_\sigma = \alpha\, \rho(\beta_\sigma) a_\sigma\, \sigma(\alpha)^{-1}$$

and

$$b'_{\sigma,\tau} = \alpha * (\beta_\sigma (a_\sigma * \sigma(\beta_\tau)) b_{\sigma,\tau} \beta_{\sigma\tau}^{-1})$$

There is a natural group structure on H^0 but again H^1 is simply a pointed set while

$$H^{-1}(\Gamma, G' \to G) := H^0(\Gamma, \ker[G' \to G])$$

is an abelian group. These cohomological objects are functorial for the category of crossed modules and, as for usual hypercohomology theories, they are invariant under quasi-isomorphisms. Cohomology in higher degrees cannot be defined (in a functorial way) without further structures.

We observe that G can be seen as the crossed module

$$[1 \to G]$$

and we have

$$H^i(\Gamma, G) = H^i(\Gamma, 1 \to G)$$

There is an exact sequence

$$1 \to H^{-1}(\Gamma, G' \to G) \to H^0(\Gamma, G') \to H^0(\Gamma, G) \to$$

$$H^0(\Gamma, G' \to G) \to H^1(\Gamma, G') \to H^1(\Gamma, G) \to H^1(\Gamma, G' \to G)$$

with

$$H^{-1}(\Gamma, G' \to G) := H^0(\Gamma, \ker \rho)$$

The exactness has to be understood for pointed sets. In general the exact sequence cannot be continued since the H^2 does not make sense for a non commutative group.

If we denote by K the kernel of $[G' \to G]$ and by L the cokernel we have another exact sequence:

$$1 \to H^0(\Gamma, K) \to H^{-1}(\Gamma, G' \to G) \to H^{-1}(\Gamma, L) \to$$

$$H^1(\Gamma, K) \to H^0(\Gamma, G' \to G) \to H^0(\Gamma, L) \to H^2(\Gamma, K)$$

$$\to H^1(\Gamma, G' \to G) \to H^1(\Gamma, L) \to H^3(\Gamma, K)$$

where $H^{-1}(\Gamma, H) = 1$. Observe that since K is abelian the $H^i(\Gamma, K)$ are defined for all i.

If moreover the complex $G' \to G$ is quasi-isomorphic to a complex of abelian groups $B \to A$ (in a way compatible with Γ-actions) then one can define hyper-cohomology groups

$$H^i(\Gamma, G' \to G) := H^i(\Gamma, B \to A)$$

in all degrees and the above exact sequence extends indefinitely.

III.2 – Galois cohomology and abelianized cohomology

Let F be a field and let $\overline{F}$ be an separable closure. Consider an algebraic variety X defined over F then the set $X(F)$ of points with value in F is the fixed point set of $\mathrm{Gal}(\overline{F}/F)$ on $X(\overline{F})$. This is the 0-th cohomology set of the Galois group with values in $X(\overline{F})$:

$$X(F) = H^0(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

We shall also use the standard notation for Galois cohomology

$$H^*(F, X) := H^*(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

whenever defined. If G is an algebraic group over F we have also the set

$$H^1(F, X) := H^1(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

Finally, if A is an abelian algebraic group we have

$$H^i(F, X) := H^i(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

for all i.

Let G be a connected reductive group over a field F and let G_{SC} be the simply connected cover of its derived subgoup. We observe that complexes

$$[G_{SC} \to G]$$

are crossed modules for the obvious adjoint actions and hence we have at hand

$$H^*(F, G_{SC} \to G)$$

in degrees ≤ 1. But in fact such a complex is quasi-isomorphic to a complex of abelian groups:

$$[Z_{sc} \to Z]$$

where Z is the center of G and Z_{sc} is the center of G_{SC}. Since hypercohomology is invariant by quasi-isomorphisms this allows to define abelian groups

$$H^*(F, G_{SC} \to G) = H^*(F, Z_{sc} \to Z)$$

in all degrees. Another quasi-isomophic complex of abelian groups is useful: let T be a torus in G and let T_{sc} its preimage in G_{SC} then

$$H^*(F, G_{SC} \to G) = H^*(F, T_{sc} \to T)$$

We shall use the following compact notation

$$H^*_{ab}(F,G) := H^*(F, G_{SC} \to G)$$

Theorem III.1 – *Let G be a connected reductive group over a field F and let G_{SC} be the simply connected cover of its derived subgoup. There is a family of abelian group $H^*_{ab}(F,G)$ with natural maps*

$$H^i(F,G) \to H^i_{ab}(F,G) \quad \textit{for} \quad i \le 1$$

giving rise to a long exact sequence

$$\to H^0_{ab}(F,G) \to H^1(F,G_{SC}) \to H^1(F,G) \to H^1_{ab}(F,G)$$

Moreover, when F is a local field

$$H^1(F,G) \to H^1_{ab}(F,G)$$

is surjective and even bijective if F is non-archimedean.

Proof: . The surjectivity follows from the existence of fundamental tori and that such tori have a vanishing H^2. The injectivity follows from a theorem due to Kneser quoted below. □

Theorem III.2 – *When F is a non-archimedean local field then*

$$H^1(F,G_{SC}) = 1$$

We observe that using the universal property of simply connected spaces one can show that the abelianization map

$$H^1(F,G) \to H^1_{ab}(F,G)$$

is functorial in G (cf.[Lab3] lemme I.6.3)

Before the introduction of abelianized cohomology by Borovoi [Bo] (see also [Lab3]), Kottwitz had found a substitute for the abelianization map but that was more subtle to define, not obviously functorial and was restricted to reductive groups over local fields. Given G reductive over some field F consider the abelian group

$$\pi_0(Z(\check{G})^\Gamma)$$

where $\check{G}$ is the complex dual group, $Z(\check{G})$ its center, Γ the Galois group $\mathrm{Gal}(\overline{F}/F)$ and π_0 the group of connected components. Now, using Tate-Nakayama duality, Kottwitz has shown that when F is a local field there exists a canonical map

$$H^1(F,G) \to \pi_0(Z(\check{G})^\Gamma)^D$$

where the exponent D denotes the Pontryagin duality. Using abelianized cohomology one recovers Kottwitz's map using that

$$[\check{G} \to (G_{SC})\check{}\,]$$

is quasi-isomorphic to

$$[Z(\check{G}) \to 1]$$

and by Tate-Nakayama duality one gets an injective homomorphism

$$H^1_{ab}(F, G) \to \pi_0(Z(\check{G})^\Gamma)^D$$

which is bijective when F is non archimedean ([Lab3] Proposition 1.7.3).

We still have to introduce the Kottwitz signs [Ko1]. Given a reductive group G consider its quasisplit inner form G^* and G^*_{ad} the adjoint group. Let $a(G)$ be the cohomology class in $H^1(F, G^*_{ad})$ defining G as an inner form of G^*. There is an isomorphism from

$$H^*_{ab}(F, G^*_{ad}) \to H^2(F, Z(G^*_{SC}))$$

and by composition we get a class

$$\overline{a}(G) \in H^2(F, Z(G^*_{SC}))$$

Now the half sum of positive roots (for some order) defines a map from $Z(G^*_{SC})$ to the group of elements of order 2 in the multiplicative group and hence if F is local we get a number $e(G) \in \{\pm 1\}$ called the Kottwitz sign. It can be shown that when $F = \mathbb{R}$ one has

$$e(G) = (-1)^{q(G^*)-q(G)}$$

where $q(G)$ is half the dimension of the symmetric space attached to G.

III.3 – Stable conjugacy and κ-orbital integrals

We shall now define stable conjugacy in general. Let G be a connected reductive group over some field F. For simplicity we assume F is a field of characteristic zero.

We denote by G_{SC} the simply connected cover of its derived subgroup. Given $\gamma \in G(F)$ we denote by I_γ the subgroup of G generated by $Z(G)$ (the center of G) and the image in G of the centralizer of γ in G_{SC}. We call I_γ the stable centralizer of γ.

When γ is strongly regular semisimple i.e. if the centralizer is a torus, then I_γ is this torus. If γ is regular semisimple i.e. if the centralizer has a torus as connected component, then again I_γ is this torus. But in general I_γ is not a torus and can even be disconnected as seen in the example of $n_0 \in SL(2,\mathbb{R})$ whose centralizer is $\pm N$ with N the group of upper-triangular unipotent matrices. Nevertheless we have a

Lemma III.3 – *Let γ be semisimple in $G(F)$. Then I_γ is a connected reductive subgroup, namely the connected component of the centralizer.*

Proof: The centralizer in G_{SC} of a semisimple element γ is a connected reductive group; this is due to Steinberg. Its image in G is again reductive and connected. It contains a maximal torus T, but any T contains $Z(G)$.

□

Consider γ and γ' in $G(F)$. We shall say that γ and γ' are stably conjugate if there is an $x \in G(\overline{F})$ such that

$$(i) \qquad x^{-1}\gamma\, x = \gamma'$$

and

$$(ii) \qquad a_\sigma = x\sigma(x)^{-1} \in I_\gamma \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{F}/F)$$

where $\mathrm{Gal}(\overline{F}/F)$ is the Galois group of $\overline{F}/F$. In other words we assume that a_σ is a 1-cochain with values in I_γ.

The same relations hold if we replace x by $y = tx$ with $t \in I_\gamma$. The second condition is automatically satisfied if the first is, whenever $G = G_{SC}$: in fact a_σ always belongs to the centralizer of γ.

The pair (x, a_σ) defines a 0-cocycle with values in the quotient set

$$I_\gamma \backslash G$$

and hence a class in the Galois cohomology set associated to this set

$$H^0(\mathrm{Gal}(\overline{F}/F), I_\gamma\backslash G)$$

also denoted

$$H^0(F, I_\gamma\backslash G) \quad \text{or} \quad (I_\gamma\backslash G)(F)$$

This is the set of rational points of the quotient $I_\gamma\backslash G$. This cohomology set is the quotient of the set of 0-hypercocycles as above by the equivalence relation:

$$(x, a_\sigma) \simeq (y, b_\sigma)$$

whenever there exists $t \in I_\gamma$ such that

$$y = tx \quad \text{and} \quad b_\sigma = t a_\sigma \sigma(t)^{-1}$$

We observe that there is an exact sequence of pointed sets

$$H^0(F, I_\gamma\backslash G) \to H^1(F, I_\gamma) \to H^1(F, G)$$

induced by the map $(x, a_\sigma) \mapsto a_\sigma$ and the inclusion $I_\gamma \to G$. One denotes by $\mathfrak{D}(F, I_\gamma\backslash G)$ the image of $H^0(F, I_\gamma\backslash G)$ into $H^1(F, I_\gamma)$ or equivalently the kernel of the next map

$$\mathfrak{D}(F, I_\gamma\backslash G) = \ker\left[H^1(F, I_\gamma) \to H^1(F, G)\right]$$

With this we have a small exact sequence of pointed sets

$$1 \to I_\gamma(F)\backslash G(F) \to (I_\gamma\backslash G)(F) \to \mathfrak{D}(F, I_\gamma\backslash G) \to 1$$

To continue, let us assume first that γ is regular semisimple and hence I_γ is a maximal torus T in G defined over F. We shall now introduce the abelian groups $\mathfrak{E}(F, T\backslash G)$. We first recall that

$$\mathfrak{D}(F, T\backslash G) = \ker\left[H^1(F, T) \to H^1(F, G)\right]$$

the group $\mathfrak{E}(F, T\backslash G)$ is an abelianized version of it namely

$$\mathfrak{E}(F, T\backslash G) = \ker\left[H^1(F, T) \to H^1_{ab}(F, G)\right]$$

There is a natural injective map

$$\mathfrak{D}(F, T\backslash G) \to \mathfrak{E}(F, T\backslash G)$$

Observe that by composition we get a map

$$(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$$

This can also be seen as follows: we have

$$\mathfrak{E}(F, T\backslash G) = \text{ Im } \left[H^1(F, T_{sc}) \to H^1(F, T)\right]$$

and that there is a natural map

$$(T\backslash G)(F) \to H^1(F, T_{sc})$$

since $H^1(F, T_{sc})$ can be identified with an abelianized avatar of

$$H^0(F, T\backslash G) = (T\backslash G)(F)$$

Assume from now on that F is a local field. When F is non archimedean the map

$$\mathfrak{D}(F, T\backslash G) \to \mathfrak{E}(F, T\backslash G)$$

is bijective since

$$H^1(F, G) \to H^1_{ab}(F, G)$$

is bijective in this case. Over the reals, as we shall see in the next section, the set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ is a subset but not usually a subgroup of $\mathfrak{E}(\mathbb{R}, T\backslash G)$.

We shall denote by $\mathfrak{K}(F, T\backslash G)$ the Pontryagin dual of the finite abelian group $\mathfrak{E}(F, T\backslash G)$. We call the elements of $\mathfrak{K}(F, T\backslash G)$ endoscopic characters. Let $\kappa \in \mathfrak{K}(F, T\backslash G)$ it defines using the map

$$(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$$

a function, again denoted κ on $(T\backslash G)(F)$. The function κ has the following multiplicative property: assume that x and $z = xy$ in G define elements in $(T\backslash G)(F)$. This means that $x\sigma(x)^{-1} \in T$ and $z\sigma(z)^{-1} \in T$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ then

$$y\sigma(y)^{-1} \in T_x := x^{-1}Tx \quad \text{and} \quad \kappa(xy) = \kappa^x(y)\kappa(x)$$

where κ^x is the character of $\mathfrak{E}(F, T_x\backslash G)$ obtained from κ via the F-isomorphism

$$T \to x^{-1}Tx$$

A κ-orbital integral for γ regular in $T(\mathbb{R})$ is defined by

$$\mathcal{O}^\kappa_\gamma(f) = \int_{(T\backslash G)(F)} \kappa(x) f(x^{-1}\gamma\, x)\, d\dot{x}$$

Implicit in the definition is a compatible choice of Haar measures on the various stable conjugates $x^{-1}Tx$ of T for $x \in (T\backslash G)(F)$ obtained by transporting invariant differential forms of maximal degree.

The stable orbital integrals are the $\mathcal{O}^1_\gamma(f)$ i.e. κ-orbital integrals when κ is trivial. They are often denoted $S\mathcal{O}_\gamma(f)$.

Now consider the case of an arbitrary semisimple element γ and let $I = I_\gamma$ its stable centralizer. We put again

$$\mathfrak{D}(F, I\backslash G) = \ker \left[H^1(F, I) \to H^1(F, G)\right]$$

it has an abelianized avatar

$$\mathfrak{E}(F, I\backslash G) = \ker \left[H^1_{ab}(F, I) \to H^1_{ab}(F, G)\right]$$

with a map

$$\mathfrak{D}(F, I\backslash G) \to \mathfrak{E}(F, I\backslash G)$$

which need not be injective nor surjective in general. Now given a character κ of $\mathfrak{E}(F, I\backslash G)$ the κ-orbital integral is defined as follows

$$\mathcal{O}^\kappa_\gamma(f) = \int_{(I\backslash G)(F)} e(I_x)\kappa(x) f(x^{-1}\gamma\, x)\, d\dot{x}$$

where $e(I_x)$ is the Kottwitz sign of the group

$$I_x := x^{-1} I\, x$$

the stable centralizer of $x^{-1}\gamma\, x$.

III.4 – Stable conjugacy and compact Cartan subgroups over $\mathbb{R}$

We shall now consider the special case where $F = \mathbb{R}$ and γ is elliptic and regular in G which means that

$$T = I_\gamma$$

is an elliptic torus i.e. such that T_{sc} is an $\mathbb{R}$-anisotropic torus.

Lemma III.4 – *Let T be an elliptic torus in a real reductive group G. For any n in $N_G(T)$ the automorphism $w = \mathrm{Ad}(n)$ restricted to T is defined over $\mathbb{R}$.*

Proof: Any automorphism w of T is defined by a $\mathbb{Z}$-linear automorphism w^* of $X^*(T_{ad})$; it suffices to prove that w^* is real. But σ the non trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts as -1 on $X^*(T_{ad})$ and hence commutes with w^*.

□

We denote by $\Omega_{\mathbb{C}}(G,T)$, or simply Ω_G, the complex Weyl group, which is the group generated by the automorphisms of T induced by $\mathrm{Ad}(n)$ with $n \in N_G(T)$ and $\Omega_{\mathbb{R}}(G,T)$, or Ω_K, the subgroup generated by the automorphisms of T induced by $\mathrm{Ad}(n)$ with

$$n \in G(\mathbb{R}) \cap N_G(T) = K \cap N_G(T)$$

where K is the maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$.

Proposition III.5 – *Let T be an elliptic torus in G over $\mathbb{R}$. There is a natural bijection*

$$\mathfrak{D}(\mathbb{R}, T\backslash G) \to \Omega_{\mathbb{C}}(G,T)/\Omega_{\mathbb{R}}(G,T)$$

Proof: Consider γ strongly regular in T and in particular $T = I_\gamma$. Then, we observe that if γ' is stably conjugate to γ then, $I_{\gamma'}$ is again an elliptic torus; but all elliptic tori are conjugate in $G(\mathbb{R})$. Hence, up to ordinary conjugacy we may replace γ' by $\gamma'' \in T$ and now there is n in the normalizer $N_G(T)$ of T in G such that

$$n^{-1}\gamma\, n = \gamma''$$

The image of n in the Weyl group is uniquely defined by the pair (γ, γ'') since γ is strongly regular but the element γ'' is defined by γ' only up to the action of the real Weyl group. This yields an injective map

$$\mathfrak{D}(\mathbb{R}, T\backslash G) \to \Omega_{\mathbb{C}}(G,T)/\Omega_{\mathbb{R}}(G,T)$$

Now the map is surjective thanks to lemma III.4.

□

More generally we observe that all conjugacy classes inside the stable conjugacy class of an elliptic element in T, intersect T.

Remark – The set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ is not usually a group. In fact $\Omega_{\mathbb{R}}(G,T)$ is not an invariant subgroup of $\Omega_{\mathbb{C}}(G,T)$ in general although $\mathfrak{D}(\mathbb{R}, T\backslash G)$ can be embedded naturally in the abelian group $\mathfrak{E}(\mathbb{R}, T\backslash G)$. For example consider $G = U(2,1)$, then

$$\Omega_{\mathbb{C}}(G,T) = \mathfrak{S}_3 \quad \text{while} \quad \Omega_{\mathbb{R}}(G,T) = \mathfrak{S}_2$$

In this case $\mathfrak{D}(\mathbb{R}, T\backslash G)$ has three elements and the group $\mathfrak{E}(\mathbb{R}, T\backslash G)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

III.5 – Discrete series and stable conjugacy

We recall that, according to Harish-Chandra, the group $G(\mathbb{R})$ has a discrete series representation if and only if there exist an $\mathbb{R}$-elliptic torus T in G. Assume from now on this is the case. They are parametrized as follows.

Given T, an $\mathbb{R}$-elliptic torus T, we consider K a maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$. Consider a Borel subgroup B_K in $K_{\mathbb{C}}$ containing T and a Borel subgroup B in G containing B_K. Now consider a character of $T(\mathbb{R})$ defined by a

$$\lambda \in X^*(T) \otimes \mathbb{C}$$

which is B-dominant. There is a discrete series representation $\pi_{\lambda+\rho}$, where ρ is half the sum of positive roots of T in B, characterized by its character given on $T(\mathbb{R})$ by

$$\Theta_{\lambda+\rho} = (-1)^q \; \frac{\sum_{w\in\Omega_K} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in\Omega_G} \varepsilon(w) e^{w\rho}}$$

where

$$q = \frac{1}{2}(\,\dim G(\mathbb{R}) - \,\dim K)$$

An L-packet of discrete series is a set of representations π_μ where

$$\mu = w(\lambda+\rho) \quad \text{with} \quad w \in \Omega_{\mathbb{C}}$$

such that μ belongs to the Weyl chamber defined by B_K. We shall denote by $\Omega(B_K)$ the subset of such w. Clearly, $\Omega(B_K)$ is a set of representatives in $\Omega_{\mathbb{C}}(G,T)$ of the quotient $\Omega_{\mathbb{R}}(G,T)\backslash\Omega_{\mathbb{C}}(G,T)$.

Proposition III.6 – *The set of representations in an L-packet of discrete series is in bijection with $\mathfrak{D}(\mathbb{R}, T\backslash G)$; the bijection depends on the choice of B.*

Proof: This follows from III.5 and the above remarks.

□

As already said the set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ embeds in the abelian group $\mathfrak{E}(\mathbb{R}, T\backslash G)$. Thus we get a pairing between the L-packet and $\mathfrak{K}(\mathbb{R}, T\backslash G)$ the Pontryagin dual of $\mathfrak{E}(\mathbb{R}, T\backslash G)$. This pairing depends on the choice of B. We shall see that this pairing has another formulation using L-groups.

Some choices of B seem to be better than others at least when G is a quasi-split group: they are those that correspond to generic representations i.e. with a Whittaker model (with respect to some further choice). For example, for $G = U(2,1)$ and $K = U(2)\times U(1)$ once B_K is chosen there are three Weyl chambers for G that are contained in the Weyl chamber defined by B_K in the Lie algebra of T. A Weyl chamber for G that contains ρ_K is a "good" choice. There is only one for $G = U(2,1)$. But for $G = SL(2)$, $\rho_K = 0$ and any choice is good.

III.6 – Pseudo-coefficients for discrete series

To simplify slightly the discussion assume that the center of $G(\mathbb{R})$ is compact. Let π_μ be a discrete series representation of $G(\mathbb{R})$ we say that a function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ is a (normalized) pseudo-coefficient for π_μ if for any **tempered** irreducible representation π we have

$$\text{trace } \pi(f) = \begin{cases} 1 & \text{if } \pi \simeq \pi_\mu \\ 0 & \text{otherwise} \end{cases}$$

Observe that it would be more canonical to consider $f\,d\mu$ where $d\mu$ is a Haar measure in $G(\mathbb{R})$ since the choice of the Haar measure enters in the definition of trace $\pi(f)$.

The existence of pseudo-coefficients is an easy consequence of the existence of multipliers due to Arthur [A] and Delorme [D]. We refer the reader to [CD] or [Lab2] for the construction. In this last paper more general functions, called index functions are constructed. They allow in particular to deal with certain linear combinations of limits of discrete series although individual ones do not have pseudo-coefficients.

A suitably normalized diagonal matrix coefficient of π_μ (or rather its complex conjugate) would satisfy these requirements but for one condition: matrix coefficients are not compactly supported unless the group is compact.

The interest of using pseudo-coefficients instead of matrix coefficients is that since they are compactly supported trace $\pi(f)$ makes sense even if π is not tempered and they can be used in the trace formula. These two properties are tied up since non tempered representation show up in the spectral decomposition of the space of square integrable automorphic forms.

We shall denote f_μ a pseudo-coefficient for π_μ, although it is highly non unique. But as regards invariant harmonic analysis this plays no role. In particular the orbital integrals are independent of the choice of the pseudo-coefficient; they are also independent of the choice of the Haar measure on $G(\mathbb{R})$ but one has to use the canonical measure on the maximal compact torus.

The orbital integrals of f_μ are easily computed for γ regular semisimple:

$$\mathcal{O}_\gamma(f_\mu) = \begin{cases} \Theta_\mu(\gamma^{-1}) & \text{if } \gamma \text{ is elliptic} \\ 0 & \text{otherwise} \end{cases}$$

where Θ_μ is the character of π_μ.

III.7 – The dual picture

Let G be a reductive connected algebraic group and let G^* be its quasi-split inner form. To G^* is associated a based root datum $(X, \Delta, \check{X}, \check{\Delta})$ with a Galois action. Then, $\check{G}$ is the **complex reductive Lie group** associated to the dual based root datum $(\check{X}, \check{\Delta}, X, \Delta)$. The Galois goup acts on the based root datum and hence on $\check{G}$ via **holomorphic automorphisms** that will be moreover assumed to preserve a splitting. The L-group attached to G is the semi-direct product

$$^LG = \check{G} \rtimes \mathbb{W}_{\mathbb{R}}$$

here $\mathbb{W}_{\mathbb{R}}$ acts via its quotient $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

A Langlands parameter is an homomorphism

$$\varphi : \mathbb{W}_{\mathbb{R}} \to {}^LG$$

whose image contains only semisimple elements and yields the identity when composed with the projection on $\mathbb{W}_{\mathbb{R}}$.

To each conjugacy class of Langlands parameter is associated an L-packet of representations of $G(\mathbb{R})$. We have explained this correspondence for $GL(2)$ and $SL(2)$ above. The general case has been established by Langlands in a paper unpublished for a long time and now available in print [Lan2]. One denotes by $\Pi(\varphi)$ the L-packet attached to φ.

Let S_φ denote the centralizer in $\check{G}$ of $\varphi(\mathbb{W}_{\mathbb{R}})$ and denote by $\mathfrak{S}_\varphi$ the quotient of S_φ by its connected component S_φ^0 times $Z(\check{G})^\Gamma$ the center of ^LG. In [She4] Diana Shelstad has established the

Proposition III.7 – *For discrete series parameters there is an isomorphism between* $\mathfrak{S}_\varphi$ *and* $\mathfrak{K}(\mathbb{R}, T\backslash G)$.

We give the proof when G is semisimple and simply connected. Consider an elliptic torus T in G. Let $\check{T}$ be the goup of complex characters of the lattice $\check{X}(T)$. Now consider a discrete series parameter φ. We may and will assume that $\check{T}$ contains $\varphi(\mathbb{C}^\times)$ the image of the subgoup $\mathbb{C}^\times \subset \mathbb{W}_{\mathbb{R}}$ but then S_φ is the subgroup of invariant elements in $\check{T}$ under the Galois action from ^LT and hence S_φ is the set of elements of order 2 in $\check{T}$. We observe that an element in $\mathfrak{K}(\mathbb{R}, T\backslash G)$ defines a complex character of $\check{X}(T)$. Hence to $\kappa \in \mathfrak{K}(\mathbb{R}, T\backslash G)$ corresponds an element $s \in \check{T}$ that moreover commutes with the Galois action. We thus get a bijective homomorphism

$$\mathfrak{K}(\mathbb{R}, T\backslash G) \to \mathfrak{S}_\varphi$$

For $\mathfrak{p}$-adic fields the Langlands parametrization and the structure of L-packets of discrete series are not known in general but it can be checked in examples that the group $\mathfrak{S}_\varphi$ is often bigger than $\mathfrak{K}$ and can even be non abelian.

III.8 – Endoscopic groups

We have to recall the Tate-Nakayama isomophism for tori. Let S be a torus defined over F and split over a Galois extension K; we denote by $\check{X}(S)$ the $\mathbb{Z}$-free module of finite rank of its cocharacters.

Theorem III.8 – *Assume that F is a local field. There is a canonical isomorphism between Tate cohomology groups*

$$\hat{H}^i(K/F, \check{X}(S)) \to \hat{H}^{i+2}(K/F, S(K))$$

In particular there is a canonical isomorphism

$$\hat{H}^{-1}(K/F, \check{X}(S)) \to H^1(K/F, S(K)) = H^1(F, S)$$

where, by definition

$$\hat{H}^{-1}(\Gamma, X) := \hat{H}_0(\Gamma, X)) = X^{N_\Gamma}/I_\Gamma X$$

where X^{N_Γ} is the kernel of the norm endomorphism of X:

$$N_\Gamma : x \mapsto \sum_{\sigma\in\Gamma} \sigma(x)$$

and I_Γ is the augmentation ideal in the group algebra $\mathbb{Z}[\Gamma]$

$$I_\Gamma = \left\{ \tau = \sum_{\sigma\in\Gamma} n_\sigma\, \sigma \;\middle|\; \sum_{\sigma\in\Gamma} n_\sigma = 0 \right\}$$

Altogether we have the

Proposition III.9 – *There is a canonical isomorphism*

$$\check{X}(S)^{N_\Gamma}/I_\Gamma \check{X}(S) \to H^1(F, S)$$

with $\Gamma = \mathrm{Gal}(K/F)$.

Now there is a surjective map

$$H^1(F, T_{sc}) \to \mathfrak{E}(F, T\backslash G)$$

and if moreover we assume T elliptic then T_{sc} is anisotropic and hence we have a bijection

$$\check{X}(T_{sc})^{N_\Gamma} \to \check{X}(T_{sc})$$

The above proposition yields a surjective map

$$\check{X}(T_{sc}) \to \mathfrak{E}(F, T\backslash G)$$

and hence it T is elliptic any character κ of $\mathfrak{E}(F, T\backslash G)$ defines a character $\overline{\kappa}$ of $\check{X}(T_{sc})$.

Let R_G and $\check{R}_G$ denote the set of roots and coroots of T in G. We observe that the coroots are homomorphisms of the multiplicative group $\mathbb{G}_m$ into T that factor through T_{sc} and hence

$$\check{R}_G \subset \check{X}(T_{sc})$$

Let us denote by $\check{R}_{\overline{\kappa}}$ the subset of coroots $\check{\alpha} \in \check{R}$ such that

$$\overline{\kappa}(\check{\alpha}) = 1$$

Lemma III.10 – *This is a (co)root system*

Proof: In fact $\check{R}_{\overline{\kappa}}$ is the root system for the connected centralizer $\check{H}$ of the image s of $\overline{\kappa}$ in $\check{T} \subset \check{G}$.

□

Let $X = X(T)$ and $\check{X} = \check{X}(T)$, there is a root datum

$$(X, R_{\overline{\kappa}}, \check{X}, \check{R}_{\overline{\kappa}})$$

which inherits, from the Galois action on T, a natural Galois action and we get a quasi-split reductive group denoted $H_{\overline{\kappa}}$ (or simply H) with maximal torus $T_H \simeq T$ and coroot system

$$\check{R}_H \simeq \check{R}_{\overline{\kappa}}$$

This is the elliptic endoscopic group attached to κ.

More generally, if we do not assume T elliptic, one can associate, as above, an endoscopic group $H_{\overline{\kappa}}$ to any characters $\overline{\kappa}$ of $\check{X}(T_{sc})/I_\Gamma\check{X}(T_{sc})$. Such $\overline{\kappa}$'s naturally occur through local-global constructions for the stabilization of the trace formula. Now $\overline{\kappa}$ defines, by restriction to $\check{X}(T_{sc})^{N_\Gamma}$, a character κ of

$$H^1(F, T_{sc})$$

which descends to a character of $\mathfrak{E}(F, T\backslash G)$ if κ is trivial on the kernel of the map

$$H^1(F, T_{sc}) \to H^1(F, T)$$

and in all cases κ defines a function on $(T\backslash G)(F)$ via the natural map

$$(T\backslash G)(F) \to H^1(F, T_{sc})$$

It gives rise, when κ is non trivial on the above kernel, to the variant of ordinary endoscopy that deals with orbital integrals weighted by a character ω as in II.3 above.

We observe that H and G share a torus and that the Weyl group $\Omega(T_H, H)$ is canonically isomorphic to a subgroup of $\Omega(T, G)$. Nevertheless there may not exist any homomorphism from H to G that extends the isomorphism $T_H \simeq T$.

Consider the image s of $\overline{\kappa}$ in $\check{T}$, its connected centralizer is $\check{H}$. An admissible embedding of ${}^L H$ in ${}^L G$ is an L-homomorphism

$$\eta : {}^L H \to {}^L G$$

that extends the natural inclusion

$$\check{H} \to \check{G}$$

(by L-homomorphism we understand an homomorphism whose restriction to $\check{H}$ is holomorphic and induces the identity on $\mathbb{W}_{\mathbb{R}}$).

A sufficient condition for the existence of an embedding is that the center of $\check{G}$ is connected. There are examples where there is no admissible embedding.

III.9 – Endoscopic transfer

Consider T an elliptic torus and κ an endoscopic character. Let H be the endoscopic group defined by (T, κ). We need some notation. We let B_G be a Borel subgroup of G containing T.

$$\Delta_B(\gamma) = \prod_{\alpha>0}(1-\gamma^{-\alpha})$$

where the product is over the positive roots for for the order defined by B. There is only one choice of a Borel subgroup B_H in H, containing T_H compatible with the isomorphism $j : T_H \simeq T$.

Proposition III.11 – *Assume there is an admissible embedding*

$$\eta : {}^L H \to {}^L G$$

One can attach to the triple (G, H, η) a character $\chi_{G,H}$ of $T(\mathbb{R})$ with the following property. Given f a pseudo-coefficient for a discrete series on G, there is a function f^H which is a linear combination of pseudo-coefficients for discrete series on H such that for a $\gamma = j(\gamma_H)$ regular in $T(\mathbb{R})$

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

where $\Delta_H^G(\gamma_H, \gamma)$ the "transfer factor" has the following expression:

$$(-1)^{q(G)+q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^{-1}) \,.\, \Delta_{B_H}(\gamma_H^{-1})^{-1}$$

We shall give, in the next section, a proof of this proposition when $G(\mathbb{R}) = U(2, 1)$ and in a further section in general. The expression for the transfer factor is borrowed from Kottwitz [Ko2].

The character $\chi_{G,H}$ depends on the choice of the admissible embedding η. To obtain an unconditional and more canonical transfer we may change a little the requirements: using the "canonical" transfer factor one gets, instead of a transfer to $H(\mathbb{R})$, a transfer to $H_1(\mathbb{R})$ some covering group of $H(\mathbb{R})$ and thus one gets functions f^{H_1} that transform according to some character of the kernel of

$$H_1(\mathbb{R}) \to H(\mathbb{R})$$

If an admissible embedding exists the character on the kernel is the restriction of some character of $H_1(\mathbb{R})$ which allows to twist the transfer so that it descends to $H(\mathbb{R})$. This will be explained in the case $G = U(2, 1)$ below.

The transfer $f \mapsto f^H$ of pseudo-coefficients can be extended to all functions in $\mathcal{C}_c^\infty(G(\mathbb{R}))$; to define it one has to extend the correspondence $\gamma \mapsto \gamma_H$, called the norm, to all semisimple regular elements and the definition of the transfer factors to all tori. This is established by Shelstad in the series of four papers [She1], [She2], [She3] and [She4].

Theorem III.12 – *Assume there is an admissible embedding*

$$\eta : {}^L H \to {}^L G$$

One can define transfer factors $\Delta_H^G(\gamma_H, \gamma)$ such that for any $f \in \mathcal{C}_c^\infty(G(\mathbb{R})$ there exists a function $f^H \in \mathcal{C}_c^\infty(H(\mathbb{R})$ with

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

whenever γ_H is a norm of γ semisimple regular and

$$S\mathcal{O}_{\gamma_H}(f^H) = 0$$

if γ_H is not a norm.

It should be observed that the correpondence

$$f \mapsto f^H$$

is not a map since f^H is not uniquely defined. In fact f^H is only prescribed through its orbital integrals. But as regards invariant harmonic analysis this is a well defined object.

The geometric transfer

$$f \mapsto f^H$$

is dual of a transfer for representations. To any admissible irreducible representation σ of $H(\mathbb{R})$ it corresponds an element σ_G in the Grothendieck group of virtual representations of $G(\mathbb{R})$ as follows. Given φ a Langlands parameter for H then $\eta \circ \varphi$ is a Langlands parameter for G^* where η is the admissible embedding

$$\eta : {}^L H \to {}^L G$$

as above. Now consider Σ the L-packet of admissible irreducible representation of $H(\mathbb{R})$ corresponding to φ and Π the L-packet of representations of $G(\mathbb{R})$, corresponding to $\eta \circ \varphi$ (that can be the empty set if this parameter is not relevant for G).

Theorem III.13 – *There is a function*

$$\varepsilon : \Pi \to \pm 1$$

such that, if we consider σ_G in the Grothendieck group defined by

$$\sigma_G = \sum_{\pi \in \Pi} \varepsilon(\pi)\, \pi$$

then $\sigma \mapsto \sigma_G$ is the dual of the geometric transfer:

$$\text{trace } \sigma_G(f) = \text{ trace } \sigma(f^H)$$

This is the Theorem 4.1.1 of [She4]. We shall prove it for discrete series of $U(2,1)$ below. We shall now give an expression for $\varepsilon(\pi)$ following section 5 of [She4].

Suppose we are given a complete set of inequivalent endoscopic groups H and for each H an admissible embedding

$$\eta : {}^L H \to {}^L G$$

Now consider a parameter

$$\varphi : \mathbb{W}_{\mathbb{R}} \to {}^L G$$

The connected centralizer of $s \in \mathfrak{S}_\varphi$ is a group $\check{H}_s$ conjugate to $\check{H}$ for some H and hence a conjugate of φ factors through $\eta({}^L H)$ and defines an L-paquet Σ_s of representations of $H(\mathbb{R})$. (The L-packet is not unique in general: it depends on the choice of the conjugate which may not be unique). On the other hand, using proposition III.7 above Shelstad defines a pairing $< s, \pi >$ between $\mathfrak{S}_\varphi$ and $\Pi(\varphi)$ and shows that

$$\varepsilon(\pi) = c(s) < s, \pi >$$

and hence the above identity

$$\sum_{\sigma \in \Sigma} \text{ trace } \sigma(f^H) = \sum_{\pi \in \Pi} \varepsilon(\pi) \text{ trace } \pi(f)$$

reads

$$\widetilde{\Sigma}_s(f^H) = \sum_{\pi \in \Pi} < s, \pi > \text{ trace } \pi(f)$$

where

$$\widetilde{\Sigma}_s(f^H) = c(s)^{-1} \sum_{\sigma \in \Sigma_s} \text{ trace } \sigma(f^H)$$

This can be inverted to give the theorem 5.2.9 of [She4]:

Theorem III.14 –

$$\text{trace } \pi(f) = \frac{1}{\#\mathfrak{S}_\varphi} \sum_{s \in \mathfrak{S}_\varphi} < s, \pi > \widetilde{\Sigma}_s(f^H)$$

IV. A SECOND EXAMPLE: $U(2,1)$

IV.1 – Discrete series and transfer for $U(2,1)$

Consider the quasisplit unitary group in three variables $G(\mathbb{R}) = U(2,1)$. A maximal compact torus $T(\mathbb{R})$ in $G(\mathbb{R})$ is such that

$$T(\mathbb{R}) \simeq U(1)^3$$

and can be represented by matrices

$$g(u,v,w) = \begin{pmatrix} u\cos\theta & 0 & iu\sin\theta \\ 0 & v & 0 \\ iu\sin\theta & 0 & u\cos\theta \end{pmatrix}$$

where u and v are complex numbers with $|u| = |v| = 1$ and $w = r(\theta)$. More precisely

$$(u,v,w) \mapsto g(u,v,w)$$

defines a twofold cover T_1 of T. The set $\mathfrak{D}(\mathbb{R},T,G)$ is isomorphic to $\mathfrak{S}_3/\mathfrak{S}_2$ and $\mathfrak{E}(\mathbb{R},T,G)$ is isomorphic to $(\mathbb{Z}_2)^2$. The root system can be represented by the $\alpha_{i,j} = \phi_i - \phi_j$ with

$$e^{i\phi_1} = ue^{i\theta} \quad e^{i\phi_2} = v \quad e^{i\phi_3} = ue^{-i\theta}$$

We take as positive roots those with $i < j$. We observe that the half sum of the roots verifies $\rho = \alpha_{1,3}$.

Choose as a maximal compact subgroup K in $G(\mathbb{R})$

$$K = \left\{ k = \frac{1}{2}\begin{pmatrix} \lambda+\nu & -\sqrt{2}\sigma & \lambda-\nu \\ \sqrt{2}\tau & 2\mu & -\sqrt{2}\tau \\ \lambda-\nu & \sqrt{2}\sigma & \lambda+\nu \end{pmatrix} \right\}$$

where k is a conjugate of

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & \tau \\ 0 & \sigma & \nu \end{pmatrix} \in U(1)\times U(2) \subset U(3)$$

It has been so chosen that it contains $T(\mathbb{R})$. The Weyl group of K is generated by w_0 the symmetry with respect to ρ.

Now, consider a dominant weight λ. One attaches to it a discrete series representation π_μ with

$$\mu = \lambda + \rho$$

The L-packet containing π_μ contains also the various $\pi_{w\mu}$. Moreover $\pi_{w\mu}$ is equivalent to $\pi_{w_0 w\mu}$ and hence we may parametrize the L-packet by the elements in the Weyl group such that $\operatorname{sign}(w) = 1$. For such a w the character of $\pi_{w\mu}$ is given by

$$\Theta_{w\mu}(\gamma) = \frac{\gamma^{w(\mu)} - \gamma^{w_0 w(\mu)}}{\gamma^\rho \Delta_B(\gamma)}$$

with

$$\Delta_B(\gamma) = \prod_{\alpha>0}(1-\gamma^{-\alpha})$$

Consider $\kappa \neq 1$ such that $\overline{\kappa}(\check{\alpha}_{1,3}) = 1$. Such a κ is unique: in fact one has necessarily

$$\overline{\kappa}(\check{\alpha}_{1,2}) = \overline{\kappa}(\check{\alpha}_{2,3}) = -1$$

The endoscopic group H one associates to κ is isomorphic to

$$U(1,1) \times U(1)$$

the positive root of T in H (for a compatible order) being

$$\alpha_{1,3} = \rho$$

The group H can be embedded in G as the subgroup of matrices in G of the form

$$g(u,v,w) = \begin{pmatrix} ua & 0 & iub \\ 0 & v & 0 \\ -iuc & 0 & ud \end{pmatrix}$$

with

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad ad - bc = 1$$

It will be useful to consider also the twofold cover

$$H_1 = U(1) \times U(1) \times SL(2)$$

(with maximal torus T_1) defined by

$$(u,v,w) \mapsto g(u,v,w)$$

Let f_μ be a pseudo-coefficient for π_μ then the κ-orbital integral of a γ regular in $T(\mathbb{R})$ is given by

$$\mathcal{O}_\gamma^\kappa(f_\mu) = \sum_{\operatorname{sign}(w)=1} \kappa(w)\Theta_\mu^G(\gamma_w^{-1})$$

but using the bijection

$$\Omega_G/\Omega_K \simeq \Omega_K\backslash\Omega_G$$

this can be rewritten

$$\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{\text{sign}\,(w)=1} \kappa(w)\Theta_{w\mu}(\gamma^{-1})$$

The transfer factor $\Delta(\gamma, \gamma_H)$ is given by

$$(-1)^{q(G)+q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^{-1})\,.\,\Delta_{B_H}(\gamma_H^{-1})^{-1}$$

for some character $\chi_{G,H}$ defined as follows. It would be canonical to take

$$\chi_{G,H}(\gamma^{-1}) = \gamma^{\rho-\rho_H}$$

but this is does not make sense since ρ_H defines a character of the twofold cover $T_1(\mathbb{R})$ but not of $T(\mathbb{R})$. A way out is to consider a weight ξ that defines a character of the cocenter of the twofold cover $H_1(\mathbb{R})$ of $H(\mathbb{R})$ so that

$$\chi_{G,H}^{-1} = e^{\rho-\rho_H+\xi}$$

descends to a character of $T(\mathbb{R})$. With such a choice we get when $\text{sign}\,(w) = 1$

$$\Delta(\gamma^{-1}, \gamma_H^{-1})\Theta^G_{w\mu}(\gamma) = -\frac{\gamma_H^{w(\mu)+\xi} - \gamma_H^{w_0 w(\mu)+\xi}}{\gamma^{\rho_H}\Delta_{B_H}(\gamma_H)}$$

We observe that $\kappa(w) = -1$ if $\text{sign}\,(w) = 1$ and $w \neq 1$ and that $w\mu$ is positive or negative with respect to B_H according to the sign of $\kappa(w)$. Hence

$$\Delta(\gamma, \gamma_H)\Theta^G_{w\mu}(\gamma^{-1}) = \kappa(w)^{-1} S\Theta^H_\nu(\gamma_H^{-1})$$

where $S\Theta^H_\nu$ is the character of the L-packet Σ_ν of discrete series for H defined by the parameter

$$\nu = w(\mu) + \xi$$

Altogether we get

$$\Delta(\gamma, \gamma_H)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_\nu S\Theta^H_\nu(\gamma_H^{-1})$$

the sum being over the $\nu = w(\mu) + \xi$ with $\text{sign}\,(w) = 1$ and this can be rewritten

$$\Delta(\gamma, \gamma_H)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_\nu S\mathcal{O}_{\gamma_H}(g_\nu)$$

where g_ν is a pseudo-coefficient for any one of the discrete series of $H(\mathbb{R})$ in the L-packet Σ_ν attached to ν. In other words we have

$$f^H_\mu = \sum_\nu g_\nu$$

Notice that the set of parameters ν depends on the choice of ξ. Alternatively we could use the canonical choice

$$\chi_{G,H}(\gamma^{-1}) = \gamma^{\rho-\rho_H}$$

if we replace H by H_1 and then the set of parameters would be canonical. This second solution is not the classical one but it seems after all more natural and at any rate considering central extensions H_1 instead of H cannot be avoided in the twisted case.

Now more generally we get

$$f^H_{w\mu} = \sum_\nu a(w,\nu) g_\nu$$

with

$$a(w_1, w_2\mu) = \kappa(w_2)\kappa(w_2 w_1)^{-1} \ .$$

Since pseudo-coefficients give a dual basis of the set of characters, in the space they generate, we get

$$\text{trace } \Sigma_\nu(f^H) = \sum_w a(w,\nu) \text{ trace } \pi_{w\mu}(f)$$

at least when f is a linear combination of pseudo-coefficients. Now, observe that

$$a(w_1, w_2\mu) = \kappa(w_2)\kappa(w_2 w_1)^{-1} = \kappa^{w_2}(w_1)^{-1}$$

Let us denote by κ_ν the character

$$x \mapsto \kappa^w(x)^{-1}$$

when $\nu = w(\mu) + \xi$. Now, using the bijection

$$\Pi_\mu \simeq \mathfrak{D}(\mathbb{R}, T, G)$$

we get a pairing denoted $<,>$ between Π_μ and $\mathfrak{K}(\mathbb{R}, T, G)$ and the transfer equation can be written

$$\text{trace } \Sigma_\nu(f^H) = \sum_{\pi\in\Pi_\mu} <\kappa_\nu, \pi> \ \text{ trace } \pi(f)$$

The above formula for the dual transfer can be rewritten

$$\text{trace } \Sigma_s(f^H) = \sum_{\pi\in\Pi_\Sigma} <s, \pi> \ \text{ trace } \pi(f)$$

where the pairing $<s,\pi>$ is the Shelstad pairing between

$$\mathfrak{S}_\varphi \simeq \mathfrak{K}(\mathbb{R}, T, G)$$

and Π_μ.

IV.2 – The dual picture for $U(2,1)$

Let us now describe the dual picture. Observe that $\overline{\kappa}$ is the image, via Tate-Nakayama duality, of

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \check{T} \subset \check{G}$$

and hence

$$\check{H} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

The holomorphic Galois action on $\check{G}$ is given by

$$g \mapsto J^t g^{-1} J^{-1}$$

with

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

while on $\check{H}$ it is $h \mapsto J_H^t h^{-1} J_H^{-1}$ with

$$J_H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

As a consequence of the difference between J and J_H there is no canonical admissible embedding of ${}^L H$ in ${}^L G$.

IV.3 – Discrete transfer

Consider a reductive Lie group $G(\mathbb{R})$ with compact maximal torus $T(\mathbb{R})$. In particular, $G(\mathbb{R})$ has discrete series. We choose an endoscopic character κ, this defines an endoscopic group H. We choose a Borel subgroup B in G containing T and a compatible Borel subgroup B_H in H. To simplify the discussion we assume that $\rho - \rho_H$ the difference of half sums of positive roots for G and H respectively defines a character of $T(\mathbb{R})$ and hence the canonical transfer factor:

$$\Delta(\gamma^{-1}) = (-1)^{q(G)-q(H)} \frac{\sum_{w\in\Omega_G} \varepsilon(w)\gamma^{w\rho}}{\sum_{w\in\Omega_H} \varepsilon(w)\gamma^{w\rho_H}}$$

is a well defined function.

Let μ be parameter for a discrete series and let f_μ be a pseudo-coefficient for π_μ. Let $w_0\rho$ be the half sum of positive roots for the Weyl chamber defined by μ i.e. $\mu = w_0\mu_0$ where μ_0 is B-dominant and regular. We have

$$(-1)^{q(G)}\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = (-1)^{q(G)} \sum_{w\in\Omega_G/\Omega_K} \kappa(w)\mathcal{O}_{\gamma_w}(\check{f}_\mu)$$

$$= \sum_{w\in\Omega_G/\Omega_K} \kappa(w) \frac{\sum_{w'\in\Omega_K} \varepsilon(w')\gamma^{ww'\mu}}{\sum_{w_1\in\Omega_G} \varepsilon(w_1)\gamma^{ww_1w_0\rho}}$$

so that

$$(-1)^{q(G)}\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = \varepsilon(w_0) \frac{\sum_{w\in\Omega_G} \kappa(w)\varepsilon(w)\gamma^{w\mu}}{\sum_{w\in\Omega_G} \varepsilon(w)\gamma^{w\rho}}$$

Hence, we have

$$(-1)^{q(H)}\Delta(\gamma^{-1})\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = \varepsilon(w_0) \frac{\sum_{w\in\Omega_G} \kappa(w)\varepsilon(w)\gamma^{w\mu}}{\sum_{w\in\Omega_H} \varepsilon(w)\gamma^{w\rho_H}}$$

Lemma IV.1 – *The function κ is left invariant under Ω_H:*

Proof: Let $w \in \Omega_G$ and $w_0 \in \Omega_H$ then the multiplicativity property of κ shows that

$$\kappa(w_0w) = \kappa^{w_0}(w)\kappa(w_0)$$

But, by definition of H, for any reflexion $s_\alpha \in \Omega_H$ one has

$$\kappa(s_\alpha) = \overline{\kappa}(\check{\alpha}) = 1 \quad \text{and} \quad \kappa^{s_\alpha} = \kappa$$

and hence, by induction on the length of w_0, one has $\kappa(w_0) = 1$ and $\kappa^{w_0}(w) = \kappa(w)$ so that

$$\kappa(w_0w) = \kappa(w)$$

□

Now introduce $\Omega_*(\mu)$ the set of representatives of the quotient $\Omega_H\backslash\Omega_G$ such that $w_*\mu$ is B_H dominant. The above lemma shows that

$$(-1)^{q(H)}\Delta(\gamma^{-1})\mathcal{O}^\kappa_\gamma(\check{f}_\mu)$$

$$= \sum_{w_*\in\Omega_*(\mu)} \kappa(w_*)\varepsilon(w_*w_0)\frac{\sum_{w\in\Omega_H}\varepsilon(w)\gamma^{ww_*\mu}}{\sum_{w\in\Omega_H}\varepsilon(w)\gamma^{w\rho_H}}$$

Altogether, if we denote by g_ν a pseudo-coefficient for the discrete series of $H(\mathbb{R})$ with parameter ν we have

$$\Delta(\gamma)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{w_*\in\Omega_*(\mu)} \kappa(w_*)\varepsilon(w_*w_0)SO_\gamma(g_{w_*\mu})$$

and hence

$$f^H_\mu = \sum_{w_*\in\Omega_*(\mu)} a(w,\mu)\, g_{w\mu}$$

with

$$a(w,\mu) = \kappa(w_*)\varepsilon(w_*w_0)$$

and where w is any element in the Ω_H-class defined by $w_*\in\Omega_*(\mu)$.

Consider $\nu = w_*\mu = w_*w_0\mu_0$ and let Σ be the L-packet of representations of $H(\mathbb{R})$ with parameter ν. We have $w_\Sigma = w_*w_0 \in \Omega_*(\mu_0)$ and hence

$$a(w_\Sigma,\mu_0) = \kappa(w_\Sigma)\varepsilon(w_\Sigma) = \kappa^{w_\Sigma}(w_0^{-1})^{-1}\kappa(w_*)\varepsilon(w_\Sigma)$$

so that

$$a(w,\mu) = \kappa^{w_\Sigma}(w_0^{-1})\, a(w_\Sigma,\mu_0)$$

Let

$$<\kappa_\Sigma,\pi> = \kappa^{w_\Sigma}(w_0^{-1})$$

where π is the discrete series for G with parameter $\mu = w_0\mu_0$; then

$$a(w,\mu) = <\kappa_\Sigma,\pi>c(\Sigma,\pi_0)$$

with

$$c(\Sigma,\pi_0) = a(ww_0,\mu_0)$$

and π_0 defined by μ_0; altogether we get

$$f^H_\pi = \sum_\Sigma <\kappa_\Sigma,\pi> c(\Sigma,\pi_0)g_\sigma$$

the sum being over L-packets Σ of discrete series for $H(\mathbb{R})$ with parameters in the orbit of μ_0 and we have chosen some $\sigma\in\Sigma$. This is equivalent to

$$c(\Sigma,\pi_0)^{-1}\ \text{trace}\ \Sigma(f^H) = \sum_\pi <\kappa_\Sigma,\pi>\ \text{trace}\ \pi(f)$$

the sum being over discrete series for $G(\mathbb{R})$ with parameters in the orbit of μ_0.

V. Further developments

V.1 – K-groups

To get a more uniform treatment of endoscopy when F is any local field it is better to introduce, following Adams-Barbasch-Vogan Kottwitz and Arthur, the K-group $\widetilde{G}$ associated to G. This is a disjoint union of groups indexed by

$$A = \text{ Im } [H^1(F, G_{SC}) \to H^1(F, G)]$$

The group G_α above $\alpha \in A$ is the inner form of G defined by α (or rather its image in the adjoint group). There are also maps between G_β and G_α:

$$\psi_{\alpha\beta} : G_\beta \to G_\alpha$$

satisfying the obvious composition rules and such that $\psi_{\alpha\beta}$ defines G_β as an inner form of G_α: we are given 1-cocycles $u_{\alpha\beta;\sigma} \in G_\alpha$ with

$$\psi_{\alpha\beta}\sigma(\psi_{\alpha\beta})^{-1} = \text{ Int } u_{\alpha\beta;\sigma}$$

and whose cohomology class belong to the image of the H^1 of the simply connected group $G_{\alpha,SC}$. When F is non archimedean one has $\widetilde{G} = G$ but for $F = \mathbb{R}$ this is not so in general.

A stable conjugacy class of $\gamma \in G_\alpha(F)$ in the K-group $\widetilde{G}$ associated to G is the set of $\gamma' \in G_\beta(F)$ for some β and such that there is $x \in G_\alpha$ with

$$\psi_{\alpha\beta}(\gamma') = x^{-1}\gamma x \quad \text{and} \quad a_\sigma = x u_{\alpha\beta;\sigma}\sigma(x)^{-1} \in I_\gamma$$

The set of classes of such 1-cocycles a build a set we denote by

$$\mathfrak{D}(F, I_\gamma\backslash\widetilde{G})$$

Assume that γ is regular semisimple, then I_γ is a torus T.

Proposition V.1 – *Let F be a local field and let T be a torus. There is a bijective map*

$$\mathfrak{D}(F, T\backslash\widetilde{G}) \to \mathfrak{E}(F, T\backslash G)$$

Proof: Recall that

$$\mathfrak{E}(F, T\backslash G) = \ker[H^1(F, T) \to H^1_{ab}(F, G)]$$

By definition of the abelianized cohomology the class of a 1-cocycle a with values in T has a trivial image in $H^1_{ab}(F, G)$ if and only if

$$a_\sigma = x u_\sigma \sigma(x)^{-1} \in T$$

for some $x \in G$ and some 1-cocycle u image of a 1-cocycle with values in G_{SC}. Hence the map is well defined and is obviously bijective.

□

V.2 – The twisted case

Another genralization is important for applications: the twisted case. We consider a group G acting on the left on a space L

$$(x, \delta) \mapsto x\,\delta \quad , \quad x \in G \qquad \delta \in L$$

which is a principal homogeneous space under this action. We say that L is a twisted G-space if we are given a G-equivariant map

$$\mathrm{Ad}_L : L \to \mathrm{Aut}\ G$$

This allows to define a right action of G on L such that

$$x\,\delta\,y = x\,(\mathrm{Ad}_L(\delta)\,y)\,\delta$$

By choosing a point $\delta_0 \in L$ we get an isomorphism

$$L \to G \rtimes \theta$$

where

$$\theta = \mathrm{Ad}_L(\delta_0)$$

but this isomorphism is not canonical when G has a non trivial center. This is why it is better to work with L rather than with $G \rtimes \theta$. One can develop as above the notion of stable conjugacy inside $L(F)$ once stable centralizers are defined: given $\delta \in L(F)$ we say that $\delta' \in L(F)$ is stably conjugate if there is $x \in G(\overline{F})$ such that

$$x^{-1}\delta x = \delta' \quad \text{and} \quad x\sigma(x)^{-1} \in I_\delta$$

for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and where I_δ is the group generated by $Z_L := Z(G)^\theta$ and the image of G_{SC}^δ the fixed points in G_{SC} under the adjoint action of δ. The stable centralizer I_δ may not be connected even if δ is regular semisimple i.e. when G_{SC}^δ is a torus.

We refer to [KS], [Lab3] and [Lab4] for a study of the twisted case in the trace formula context. As regards the real case Shelstad's results have been extended to the twisted case by Renard [Ren].

Remark – We warn the reader that the definition given for the stable centralizer in [Lab3] differs slightly from the one given above and used in [Lab4].

V.3 – Trace formula stabilization

Last but not least the main motivation for studying endoscopy is the adelic trace formula. In the old paper [LL] the trace formula for $SL(2)$ was expressed as a sum of stable trace formulas for its endoscopic groups. The stabilization gives a better understanding of the space of automorphic forms and allows to establish cases of Langlands functoriality. This was the beginning of all this game.

Later on various other instances of endoscopy and of twisted endoscopy were treated like the Base Change for $GL(2)$ by Saito-Shintani and Langlands [Lan1] and for unitary groups in three variables by Rogawski [Rog]. Endoscopy also played a role at the place where it was first discovered: for computing the Zeta function of Shimura varieties.

At the present time the problem of stabilizing the trace formula which is to express the trace formula for G (or more generally the trace formula for a twisted space) as a linear combination of stable trace formulas for its endoscopic groups is not completely solved.

Wonderful progress have been made by Jim Arthur in first establishing the trace formula for arbitrary reductive groups and then by developping techniques that have allowed him to get a conditional stabilization of the (non twisted) trace formula. The condition is about the endoscopic transfer for non archimedean fields. Observe that transfer factors have been defined for all local fields by Langland and Shelstad [LS]. One needs the existence of the transfer: given f there exists f^H satisfying the transfer identity recalled below and one needs also that the so-called fundamental lemma holds.

The fundamental lemma: *Assume that f is the characteristic function of an hyper-special maximal compact subgroup in $G(F)$ (when this makes sense) and let f^H be the similar function for H. Then the transfer identity holds:*

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

whenever γ_H is a norm of γ and it vanishes if γ_H is not a norm.

In fact one also needs a weighted variant of it. These questions have resisted the attacks until recently although many important advances have been made by many people among which I should mention Waldspurger in particular: he has proved that the non archimedean transfer holds if the fundamental lemma is true.

A proof of the fundamental lemma for unitary groups has been obtained last year by Laumon and Ngô Bao Châu. But the weighted version of it is still lacking to finish the stabilization even for unitary groups and of course one needs the twisted analogues if one wants to deal with the twisted case.

Things are now progressing faster. For example recent progress have been made by Waldspurger to reduce the twisted case to the ordinary one.

As a last word I would like to say that endoscopy is not the final word and some are looking

beyond endoscopy

but this is another story.

VI. Bibliography

References

[Bo] **M. V. Borovoi**, *Abelian Galois cohomology of Reductive groups*, Mem. Amer. Math. Soc. **132** (1998), No. 626.

[A] **J. Arthur**, *A Paley-Wiener theorem for real reductive groups*, Acta Math. **150** (1983), 1-89.

[CD] **L. Clozel P. Delorme**, *Pseudo-coefficients et cohomologie des groupes réductifs*, C.R. Acad. Sci. Paris **300** No. 11 (1985), 385-387.

[D] **P. Delorme**, *Multipliers for the convolution algebra of left and right K-finite compactly supported smooth functions on a semi-simple Lie group*, Invent. Math. **75** (1984), 9-23.

[Ko1] **R. Kottwitz**, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), p. 289-297.

[Ko2] **R. Kottwitz**, *Shimura Varieties and λ-adic Representations*, Perspectives in Math. **10** Academic Press (1988), pp. 161-210.

[KS] **R. Kottwitz D. Shelstad**, *Foundations of Twisted Endoscopy*, Astérisque **255** (1999).

[Lab1] **J.-P. Labesse**, *Cohomologie, L-groupes et fonctorialité*, Compositio **55** (1985), p. 163-184.

[Lab2] **J.-P. Labesse**, *Pseudo-coefficients très cuspidaux et K-théorie*, Math. Ann. **291** (1991), p. 607-616.

[Lab3] **J.-P. Labesse**, *Cohomologie, Stabilisation et changement de base* , Astérisque **257** (1999).

[Lab4] **J.-P. Labesse**, *Stable twisted trace formula: elliptic terms*, Journ. Inst. Math. Jussieu **3** (2004), p. 473-530.

[LL] **J.-P. Labesse R.P. Langlands**, *L-indistinguishability for SL_2*, Can. J. Math. **31** (1979), p. 726-785.

[Lan1] **R.P. Langlands**, *Base Change for $GL(2)$*, Annals of Math. Studies **96** Princeton Univ. Press (1980).

[Lan2] **R.P. Langlands**, *On the classification of irreducible representations of real algebraic groups*, Math. Surveys and Monographs **31** AMS (1989). p 101-170

[LS] **R.P. Langlands D. Shelstad**, *On the Definition of Transfer Factors*, Math. Annalen **278** (1987). p 219-271

[Ren] **D. Renard**, *Twisted endoscopy for real groups*, Journ. Inst. Math. Jussieu **2** (2003), p. 529-566.

[Rog] **J. Rogawski**, *Automorphic Representations of Unitary Groups in Three Variables*, Annals of Math. Studies **123** (1990).

[She1] **D. Shelstad**, *Characters and inner forms of a quasi-split group over $\mathbb{R}$*, Compositio Math. **39** (1979), p. 11-45.

[She2] **D. Shelstad**, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. E.N.S. **12** (1979), p. 1-31.

[She3] **D. Shelstad**, *Embeddings of L-groups*, Can. J. Math. **33** (1981), p. 513-558.

[She4] **D. Shelstad**, *L-Indistinguishability for Real Groups*, Math. Annalen **259** (1982), p. 385-430.