Finite Difference Equations

- As a warmup consider the Fibonacci Sequence:

\[ a_0 = a_1 = 1, \quad a_{n+2} = a_{n+1} + a_n, \quad n = 0, 1, 2, \ldots \]
Finite Difference Equations

• A linear finite difference (LFDE) equation is of the form

\[ \sum_{j=0}^{k} \alpha_j y_{n+j} = r_n, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (1)

• We’ll assume that

\[ \alpha_k \alpha_0 \neq 0 \] \hspace{1cm} (2)

• These are infinitely many equations in infinitely many variables

\[ y_0, y_1, y_2, \ldots \]

• The \( \alpha_j \) are the coefficients of the LFDE. The \( r_n \) are inelegantly called the right hand sides.

• If \( r_n = 0 \) for all \( n = 0, 1, 2, \ldots \) the LFDE is said to be homogeneous:

\[ \sum_{j=0}^{n} \alpha_j y_{n+j} = 0, \quad n = 0, 1, 2, \ldots \] \hspace{1cm} (3)

\[ \text{Before we get deeper into it: The general solution of (1) equals any particular solution of (1) plus the general solution of the homogeneous LFDE (3).} \]
• The solutions of (3) form a linear space.
• The dimension of the solution space of (3) is $k$. 
• The general solution of (1) equals any particular solution of (1) plus the general solution of the homogeneous LFDE (3).
• Definition: The polynomial

\[ p(t) = \sum_{j=0}^{k} \alpha_j t^j \]

is the \textbf{characteristic polynomial} of the LFDE (1):

\[ \sum_{j=0}^{n} \alpha_j y_{n+j} = r_n, \quad n = 0, 1, 2, \ldots \]
• If

$$p(r) = 0$$

then

$$y_n = r^n$$

is a solution of the homogeneous LFDE (3):

$$\sum_{j=0}^{n} \alpha_j y_{n+j} = 0, \quad n = 0, 1, 2, \ldots \quad (4)$$

• If the roots of $p$ are distinct then we get $k$ linearly independent solutions of the form (4).
• Examples:
• If the roots are **multiple**, say,

\[ p(r) = p'(t) = \ldots = p^{(\mu-1)}(r) = 0, \quad p^{(\mu)}(r) \neq 0 \]

then the sequences

\[ n^j r^n, \quad j = 0, \ldots, \mu - 1, \quad n = 0, 1, 2, \ldots \]

are linearly independent solutions of the homogeneous problem (3).
• Thus we can find the general solution of the homogeneous problem if we know the roots of the characteristic polynomial and their multiplicities.

• Finding a particular solution of the inhomogeneous problem may be hard.

• Example

• Find a particular solution of

\[ \sum_{j=0}^{k} \alpha_j y_{n+j} = 1 \]

assuming that \( p(1) = 0 \).