Fundamentals of Stability

In this note, we examine

A simple Model of Error Propagation.

Suppose we have a stepwise procedure where the global (i.e., accumulated) error at each step is denoted by $e_n$. We assume that the error propagation is defined by

$$e_0 = 0 \quad \text{and} \quad e_{n+1} = \gamma e_n + \varepsilon, \quad n = 0, 1, 2, \ldots$$

(1)

Here, $\varepsilon$ is the local error, $e_n$ is the global error (after $n$ steps), and $\gamma$ is the amplification factor. We also refer to $e_n$ simply as the error.

Note. Major simplifying assumptions are: The local error and the amplification factor are constant, and there is only one component of the error propagating in this fashion.

Note. We will encounter several variants of local and global errors.

Example, Quadrature. In quadrature the local errors simply add up, and so the amplification factor is 1.
Example, Euler’s Method. Suppose we solve the initial value problem

\[ y' = y, \quad y(0) = 1 \quad (2) \]

which obviously has the solution \( y(x) = e^x \). If we apply Euler’s method we obtain

\[ y_{n+1} = (1 + h)y_n \quad \text{and} \quad e_n = e^{xn} - y_n. \quad (3) \]

In this case,

\[ x_n = nh, \quad y(x_n) = e^{nh}, \quad y_n = (1 + h)^n. \]

and

\[ e_{n+1} = (1 + h)e_n + e^{xn} \sum_{i=2}^{\infty} \frac{h^i}{i!}. \]

To see this note that

\[
e_{n+1} = e^{xn+1} - (1 + h)^{n+1} \\
= e^{xn} + he^{xn} - (1 + h)^{n+1} + e^h e^{xn} - e^{xn} - he^{xn} \\
= (1 + h)(e^{xn} - (1 + h)^n)) + (e^h - (1 + h)) e^{xn} \\
= (1 + h)e_n + (e^h - (1 + h)) e^{xn} \\
= (1 + h)e_n + e^{xn} \sum_{i=2}^{\infty} \frac{h^i}{i!}
\]
Thus the error propagation in this case is of the form (1), except that the local error is slowly growing with $n$.

It is straightforward to verify that

$$e_n = (1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}) \varepsilon = \begin{cases} n\varepsilon & \text{if } \gamma = 1 \\ \frac{1 - \gamma^n}{1 - \gamma} \varepsilon & \text{else} \end{cases}$$

(4)

We ask what happens to the error as $n$ tends to infinity. There are four cases to consider:

- $|\gamma| < 1$
  
  In the limit, the global error is proportional to the local error:
  
  $$\lim_{n \to \infty} e_n = \frac{1}{1 - \gamma} \varepsilon.$$  (5)

- $\gamma = 1$
  
  The error grows indefinitely but slowly:
  
  $$e_n = n\varepsilon.$$  (6)

- $\gamma = -1$
  
  The error alternates (between 0 and $\varepsilon$).
  
  This case is of no great practical interest.
The error is dominated by the exponentially growing term $\gamma^{n-1}\varepsilon$.

We now make our assumptions more realistic, contemplating a situation where we take increasingly many smaller and smaller steps to cover a given distance. As the step size decreases, the local error decreases also. Formally, our assumptions are

$$\varepsilon = h^{p+1}, \quad h = \frac{1}{n},$$

and we ask about the limits as

$$h \to 0, \quad n \to \infty.$$ (8)

**Note.** The number $p$ is usually an integer, and called the order of the method.

**Note.** More realistically, the amplification factor also depends on $h$ (and changes from step to step). There usually will be several superimposed process like the one we consider. We could also introduce a constant factor multiplying the local error. This would add no insights, however.

**Example.** In the above example, the order of Euler’s method is 1, and the local error is $-\frac{h^2}{2} y_n +$ higher order terms.

Substituting in (5) we obtain

$$e(h) = e_{1/h} = \begin{cases} \frac{1}{h} h^{p+1} & \text{if } \gamma = 1 \\ \frac{1 - \gamma^{1/h}}{1 - \gamma} h^{p+1} & \text{else} \end{cases}$$ (9)
We ask what happens as $h$ tends to zero, and consider the same four cases as before.

$|\gamma| < 1$ Clearly

$$e(h) = O\left(h^{p+1}\right). \quad (11)$$

In this cases, the global error is of the same order as the local error.

$\gamma = 1$ We obtain

$$e(h) = \left(\frac{1}{h}\right) h^{p+1} = h^p. \quad (12)$$

The global error is of order one less than the local error. This is precisely the situation we encountered with quadrature.

$\gamma = -1$ In this cases $e_n$ oscillates between zero and $h^{p+1}$. The global error is of the same order

\footnote{A function $\phi(h)$ is $O\left(h^{p+1}\right)$ if there is a constant $C$ such that

$$\lim_{h \to 0} \frac{\phi(h)}{h^{p+1}} = C \neq 0.$$}
as the local error. However, this case is of no great practical interest.

|\gamma| > 1

In this cases \(e(h)\) grows unboundedly, and exponentially, as \(h \to 0\). To see this observe

\[
e(h) = \frac{1 - \gamma^{1/h}}{1 - \gamma} h^{p+1}
= \frac{1}{1 - \gamma} h^{p+1} - \frac{\gamma^{1/h}}{1 - \gamma} h^{p+1}.
\] (13)

The first term clearly goes to zero. For the second term we can ignore the constant factor \(1/(1 - \gamma)\). The remaining term is

\[
\gamma^{1/h} h^{p+1} = \frac{\gamma^n}{n^{p+1}}.
\] (14)

We are interested in the behavior of this expression as \(n\) tends to infinity. Applying the rule of L’Hôpital \(p + 1\) times we eventually get \((p + 1)!\) in the denominator, and \((\log \gamma)^{p+1} \gamma^n\) in the numerator. Thus the global error grows exponentially.

**Conclusions**

In this simple example, we encountered the following major issues in solving IVPs of ODEs:
1. The concept of a *fixed station limit* where the number of steps goes to infinity and the size of the individual steps goes to zero while the distance covered remains constant.

2. The concept of *error propagation* where the error generated by the \( n \)-th step depends on the entire preceding history, not just the \( n \)-th step-size.

3. We observed that with an amplification factor equal to 1 the order of the global error is one less than that of the local error. (We will see later that there must be one component of the error with an amplification factor equal to 1 if we wish to obtain any meaningful results at all. Thus the order of the global error will *always* be one less than that of the local error.)

4. We also observed that if the absolute value of the amplification factor is greater than 1 then the global error will grow exponentially as we decrease the step size. This phenomenon is called *instability*.

5. If the amplification factor is negative we obtain oscillating errors. Otherwise we obtain exponentially growing or decaying errors. (Actually, we will also encounter *complex* amplification factors in which case the error will exhibit a low frequency oscillation with a superimposed exponential growth or decay.)
Exercise

Consider the error propagation in (1) for the more realistic case

\[ h = \frac{1}{n}, \quad \varepsilon = h^{p+1}, \quad \text{and} \quad \gamma = 1+h \quad \text{where} \quad h > 0 \]

and examine the behavior of \( e(h) \) as \( h \to 0 \).