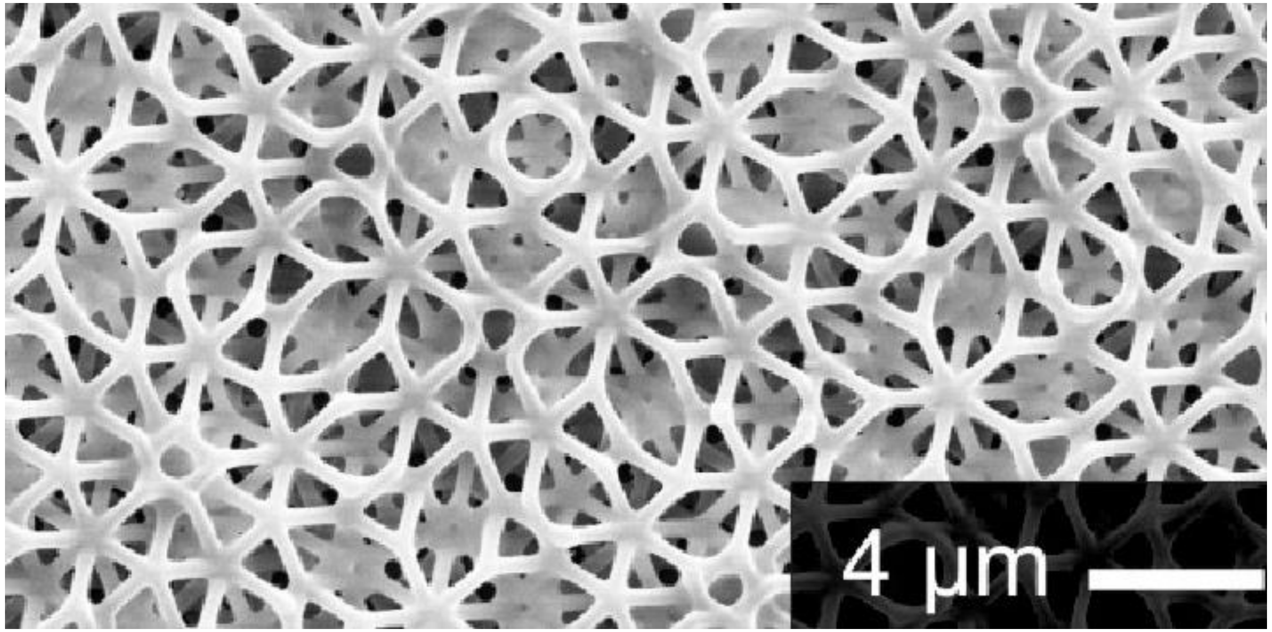
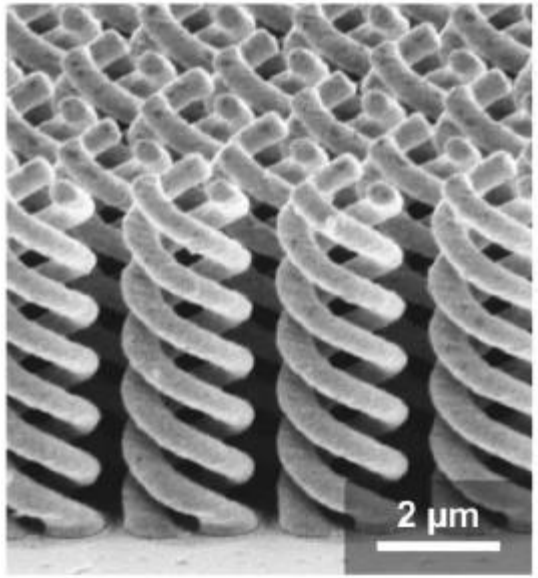
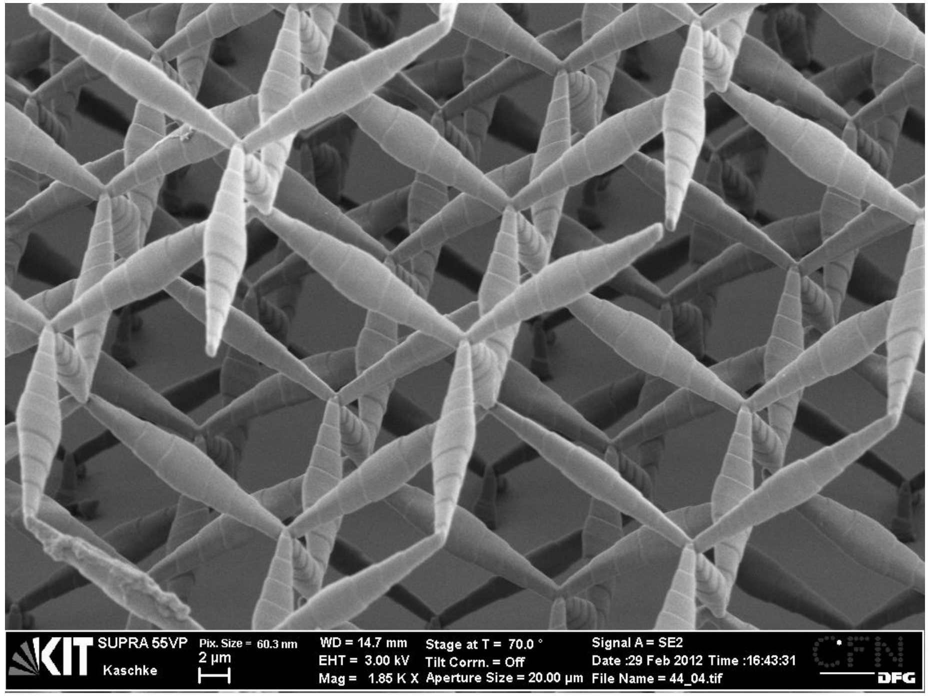
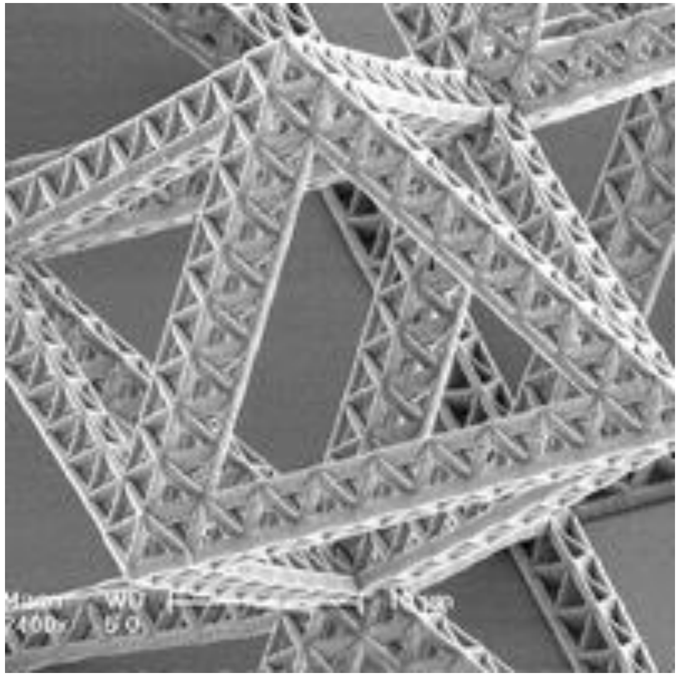
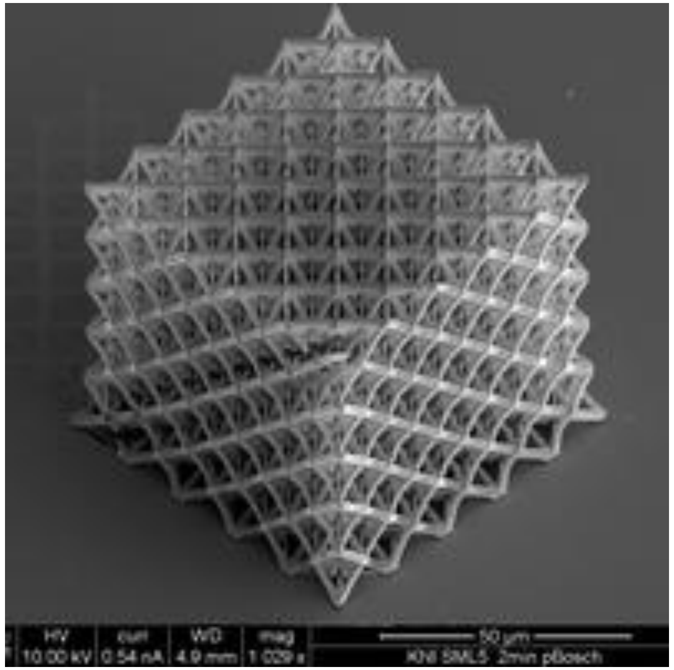


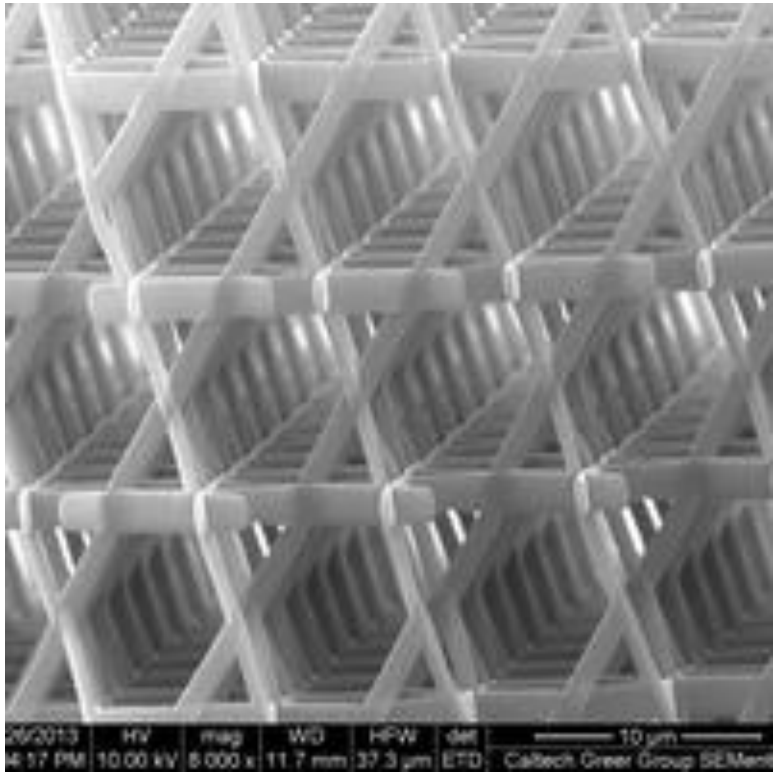




Group of
Martin Wegener

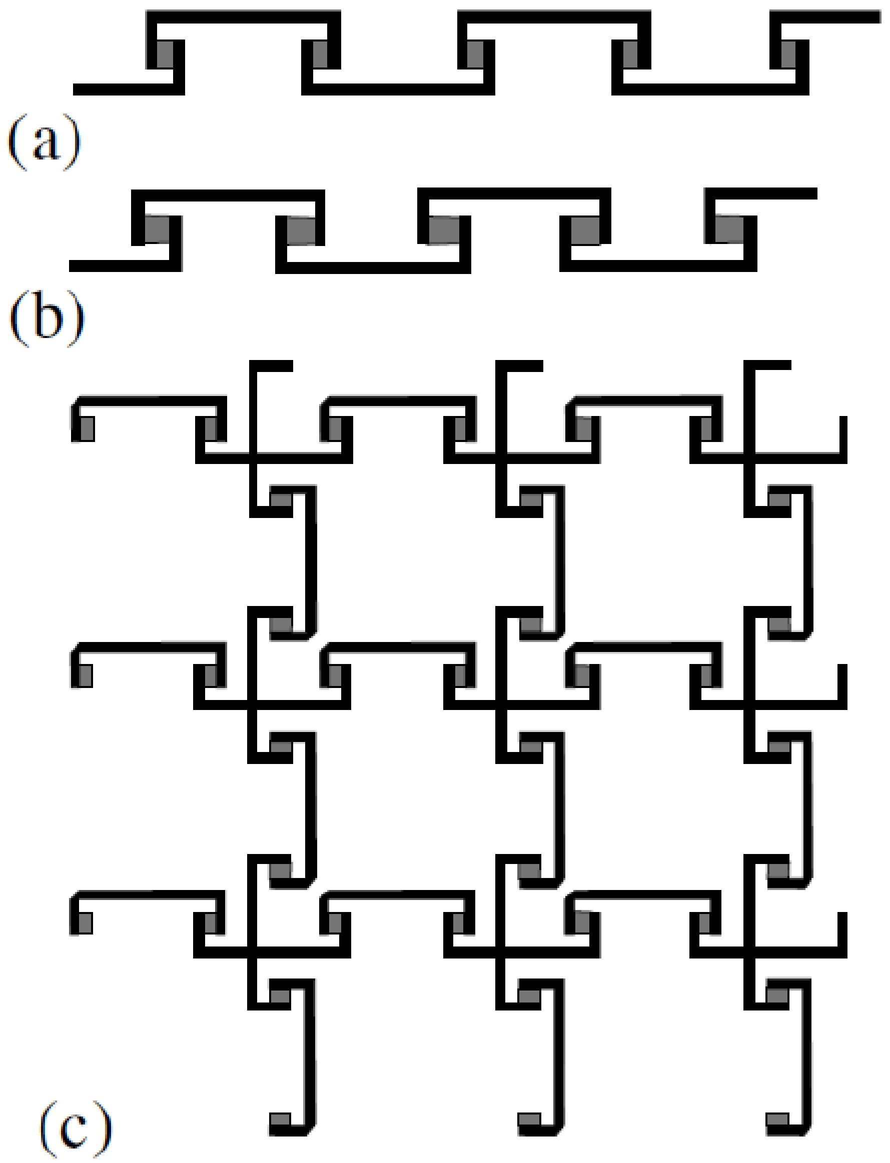






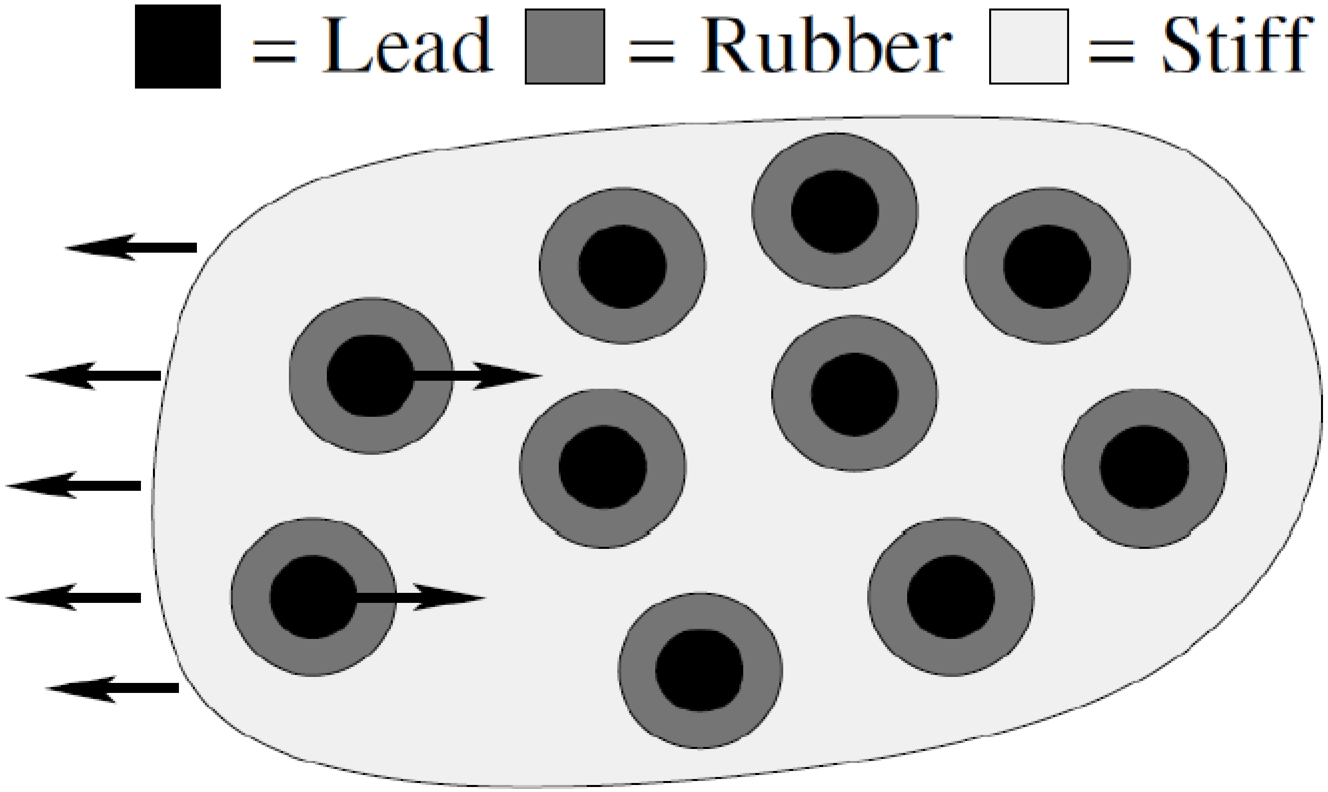
Group of Julia Greer

Another example: negative expansion from positive expansion

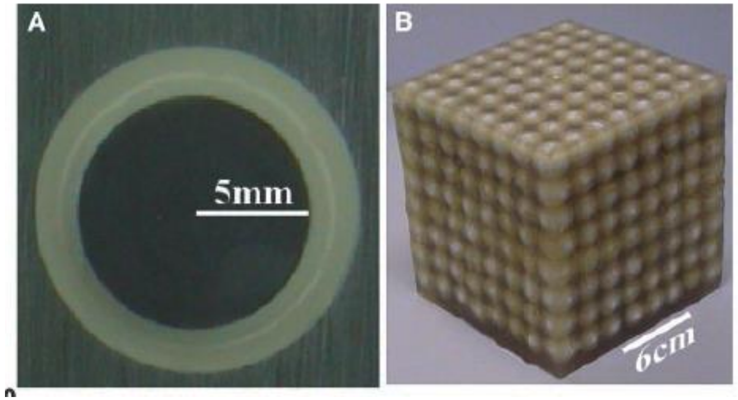


Original designs: Lakes (1996); Sigmund & Torquato (1996, 1997)

Sheng, Zhang, Liu, and Chan (2003) found that materials could exhibit a negative effective density over a range of frequencies

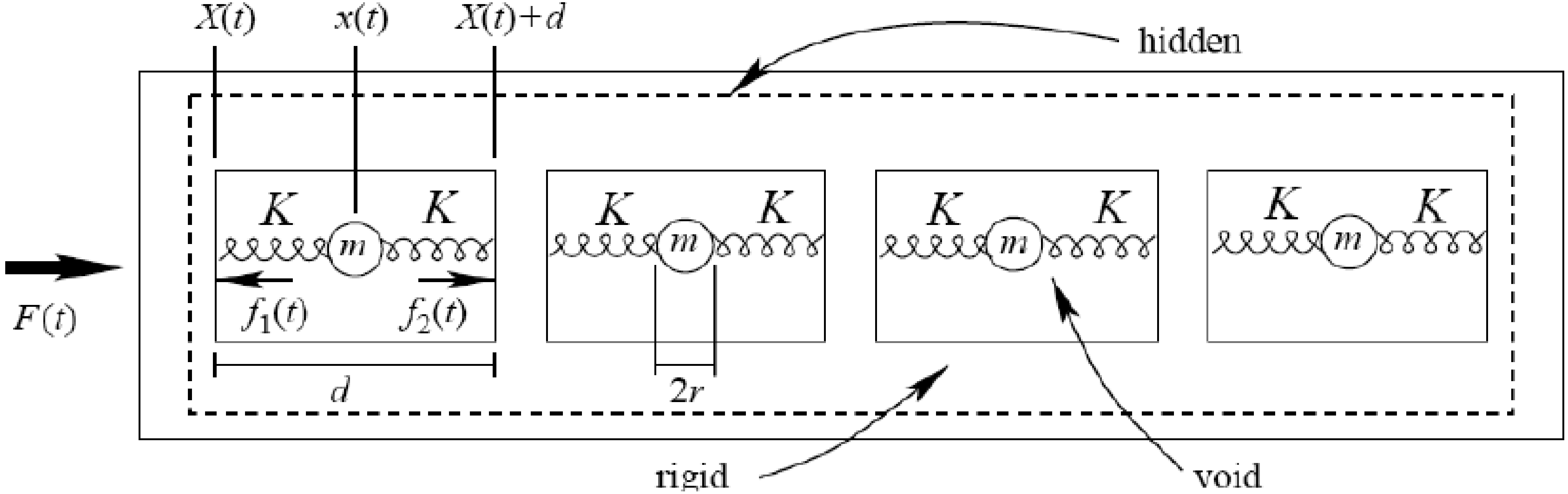


Experiment: Liu et. al (2000)



Mathematically the observation goes back to Zhikov (2000) also Bouchitte & Felbacq (2004)

A simplified one-dimensional model:



$$\hat{P} = M\hat{V}, \quad \text{with} \quad M = M_0 + \frac{2Knm}{2K - m\omega^2},$$

(With John Willis)

Early work recognizing anisotropic and negative densities

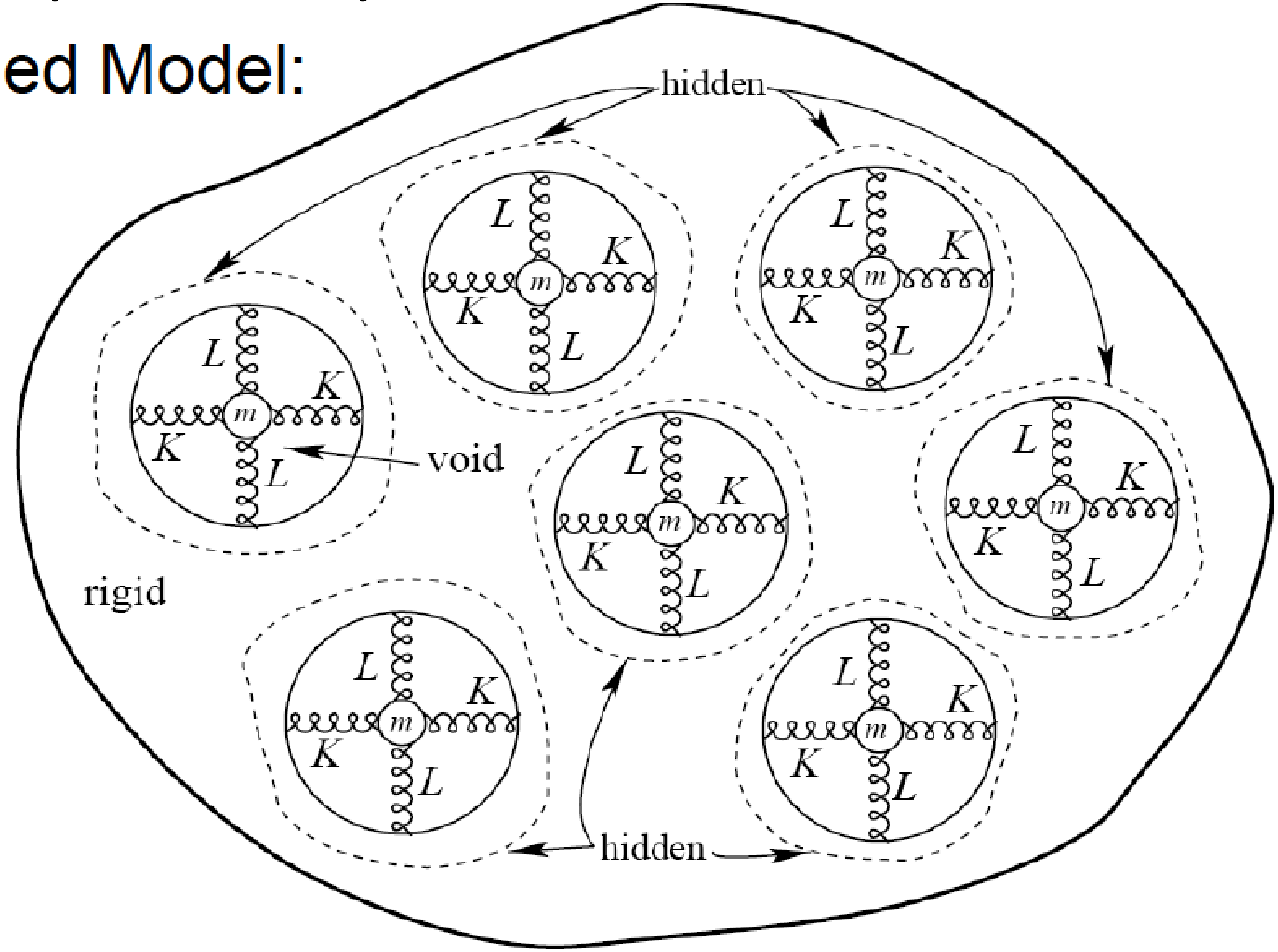
Aurialt and Bonnet (1984, 1995)

“The monochromatic macroscopic behavior is elastic, but with an effective density ϱ^{eff} of tensorial character and depending on the pulsation”

"hatched areas correspond to negative densities ϱ^{eff},
i.e., to stopping bands."

Anisotropic Density

Simplified Model:

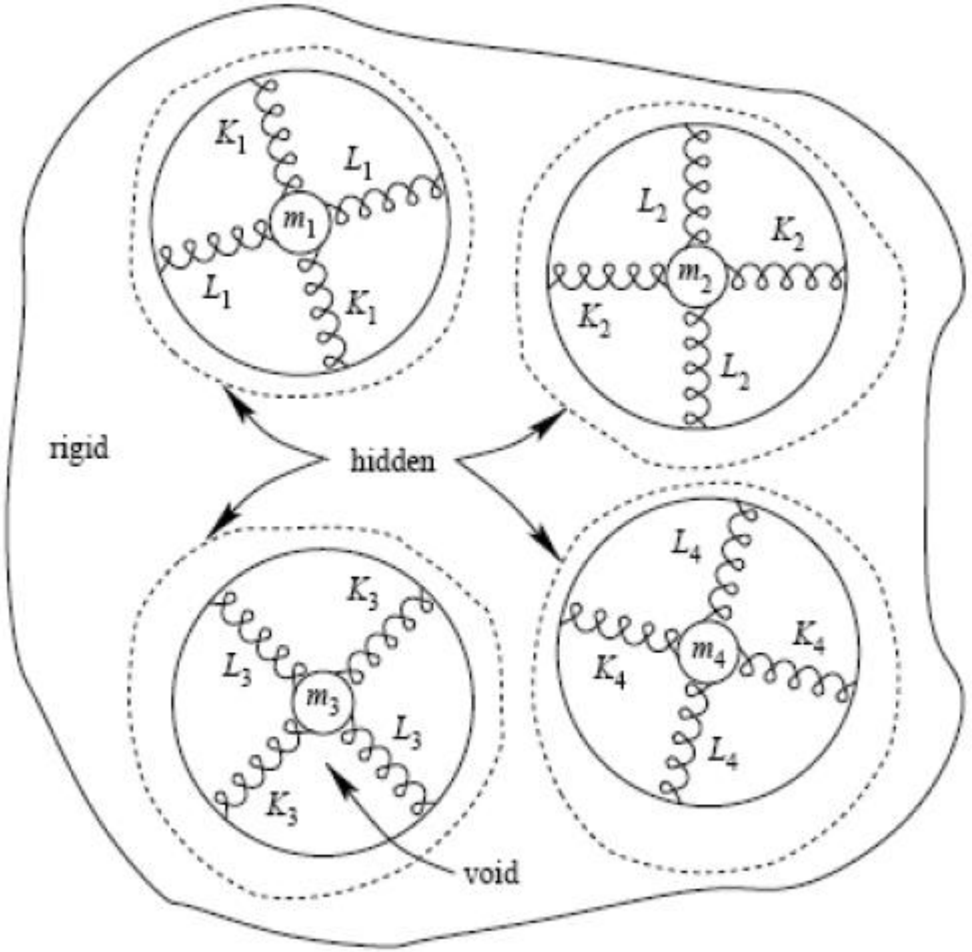


Anisotropic density in layered materials: Schoenberg and Sen (1983)

The springs could have some damping in which case the mass will be complex

(With John Willis)

Seemingly rigid body



Eigenvectors of the effective mass density can rotate with frequency

(With John Willis)

What do we learn?
For materials with microstructure, Newton's law

$$F = ma$$

needs to be replaced by

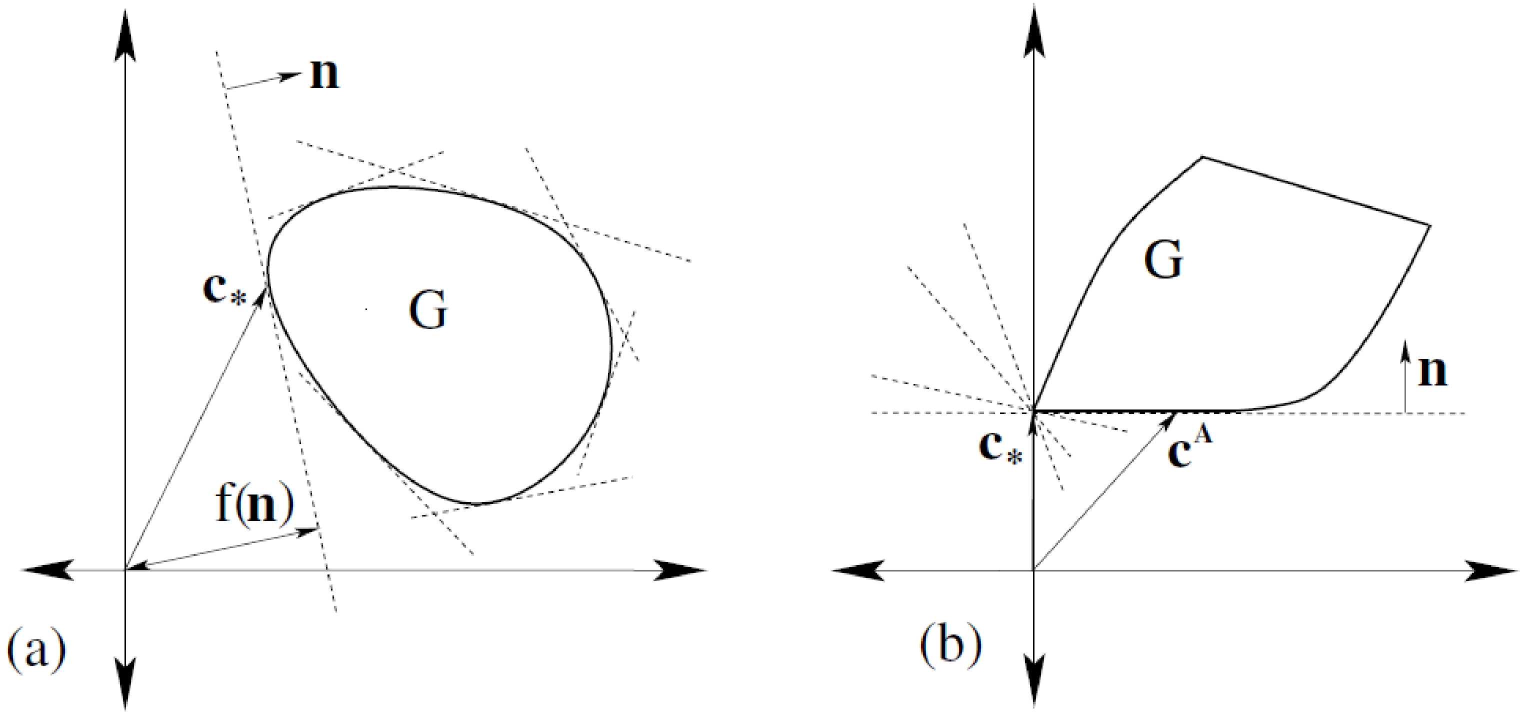
$$F(t) = \int_{-\infty}^{t} K(t' - t)a(t')\, dt'$$

It takes some time for the internal masses to respond to the macroscopically applied force.

(With John Willis)

Recall: A convex set G can be characterized by its Legendre transform:

$$f(\mathbf{n}) = \min_{\mathbf{c} \in G} \mathbf{n} \cdot \mathbf{c}.$$



G-closures are not convex sets but can be characterized by their W-transform

$$W_f(\mathbf{N},\mathbf{N}') = \min_{\mathbf{C}_*\in GU_f} (\mathbf{C}_*,\mathbf{N}) + (\mathbf{C}_*^{-1},\mathbf{N}'),$$

$$(\mathbf{N},\mathbf{C}) = N_{ijk\ell}C_{ijk\ell}$$

$$\bigcap_{\substack{\mathbf{N},\mathbf{N}'\geq 0 \\ \mathbf{N}\mathbf{N}'=0}} \{\mathbf{C} : (\mathbf{C},\mathbf{N}) + (\mathbf{C}^{-1},\mathbf{N}') \geq W_f(\mathbf{N},\mathbf{N}')\} = GU_f.$$

W-transforms generalize the idea of Legendre transforms

$$\mathbf{N} = \sum_{i=1}^{2} \boldsymbol{\epsilon}_i^0 \otimes \boldsymbol{\epsilon}_i^0, \quad \mathbf{N}' = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 \otimes \boldsymbol{\sigma}_j^0,$$

Need to know the 7 energy functions

$$
\begin{aligned}
W_f^0(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\boldsymbol{\sigma}_6^0) &= \min_{\mathrm{C}_*\in GU_f}\sum_{j=1}^{6}\boldsymbol{\sigma}_j^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0,\\
W_f^1(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\epsilon_1^0) &= \min_{\mathrm{C}_*\in GU_f}\left[\epsilon_1^0:\mathrm{C}_*\epsilon_1^0+\sum_{j=1}^{5}\boldsymbol{\sigma}_j^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right],\\
W_f^2(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\epsilon_1^0,\epsilon_2^0) &= \min_{\mathrm{C}_*\in GU_f}\left[\sum_{i=1}^{2}\epsilon_i^0:\mathrm{C}_*\epsilon_i^0+\sum_{j=1}^{4}\boldsymbol{\sigma}_j^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right],\\
W_f^3(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\epsilon_1^0,\epsilon_2^0,\epsilon_3^0) &= \min_{\mathrm{C}_*\in GU_f}\left[\sum_{i=1}^{3}\epsilon_i^0:\mathrm{C}_*\epsilon_i^0+\sum_{j=1}^{3}\boldsymbol{\sigma}_j^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right],\\
W_f^4(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\epsilon_1^0,\epsilon_2^0,\epsilon_3^0,\epsilon_4^0) &= \min_{\mathrm{C}_*\in GU_f}\left[\sum_{i=1}^{4}\epsilon_i^0:\mathrm{C}_*\epsilon_i^0+\sum_{j=1}^{2}\boldsymbol{\sigma}_j^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right],\\
W_f^5(\boldsymbol{\sigma}_1^0,\epsilon_1^0,\epsilon_2^0,\epsilon_3^0,\epsilon_4^0,\epsilon_5^0) &= \min_{\mathrm{C}_*\in GU_f}\left[\left(\sum_{i=1}^{5}\epsilon_i^0:\mathrm{C}_*\epsilon_i^0\right)+\boldsymbol{\sigma}_1^0:\mathrm{C}_*^{-1}\boldsymbol{\sigma}_1^0\right],\\
W_f^6(\epsilon_1^0,\epsilon_2^0,\epsilon_3^0,\epsilon_4^0,\epsilon_5^0,\epsilon_6^0) &= \min_{\mathrm{C}_*\in GU_f}\sum_{i=1}^{6}\epsilon_i^0:\mathrm{C}_*\epsilon_i^0.
\end{aligned}
$$

Orthogonality conditions

$$(\boldsymbol{\epsilon}_i^0, \boldsymbol{\sigma}_j^0) = 0, \qquad (\boldsymbol{\epsilon}_i^0, \boldsymbol{\epsilon}_k^0) = 0, \qquad (\boldsymbol{\sigma}_j^0, \boldsymbol{\sigma}_\ell^0) = 0$$

for all i, j, k, ℓ with $i \neq j,\ i \neq k,\ j \neq \ell.$

Result of Avellaneda (1987): If $\mathbf{C}_1 \geq \mathbf{C}_2$

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

$$W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{i=1}^{6} \boldsymbol{\epsilon}_i^0 : \mathbf{C}_* \boldsymbol{\epsilon}_i^0.$$

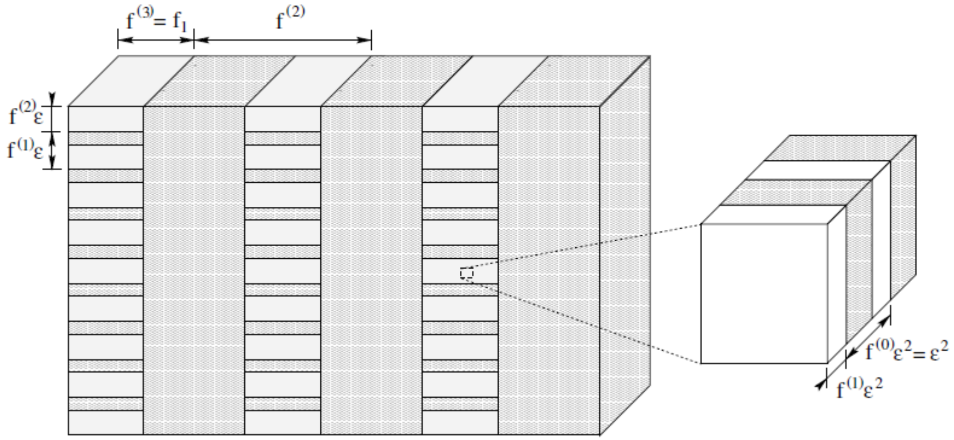
can be easily computed

They are attained by sequentially layered laminates, and we call the material which attains the minimum in

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

the Avellaneda material, with elasticity tensor

$$\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0)$$



Maxwell (1873)

Obvious bounds:

$$
\begin{aligned}
\sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\epsilon}_1^0), \\
\sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0), \\
\sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0), \\
\sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0), \\
\boldsymbol{\sigma}_1^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0 &\leq W_f^5(\boldsymbol{\sigma}_1^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0), \\
0 &\leq W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0).
\end{aligned}
$$

Main result: in many cases these bounds are sharp

Theorem

$$
\begin{aligned}
\lim_{\delta\to 0} W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0) &= \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0, \\
\lim_{\delta\to 0} W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0) &= \sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0, \\
\lim_{\delta\to 0} W_f^5(\boldsymbol{\sigma}_1^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0) &= \boldsymbol{\sigma}_1^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0, \\
\lim_{\delta\to 0} W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) &= 0.
\end{aligned}
$$

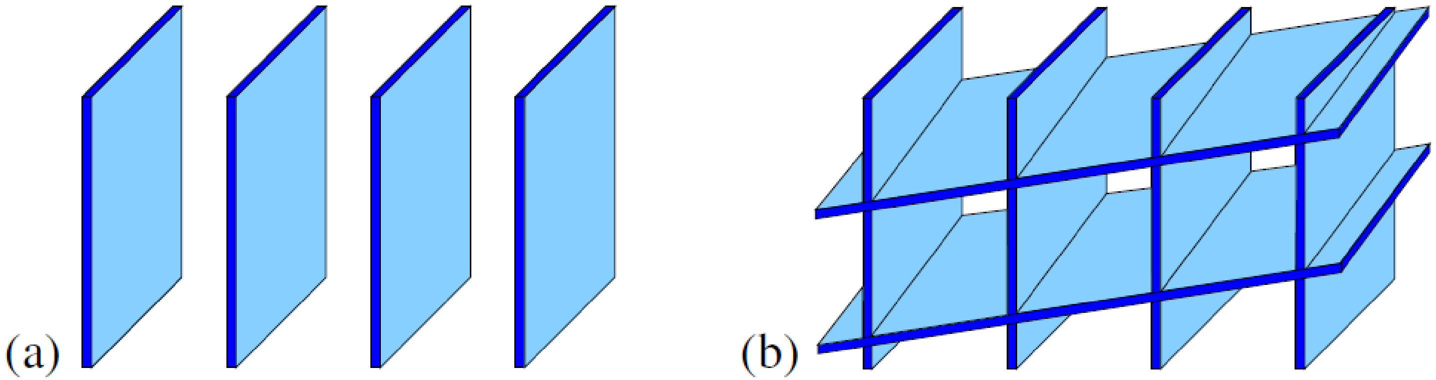
When ϵ_1^0 has one zero eigenvalue, and the other eigenvalues of opposite signs,

$$W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \epsilon_1^0) = \sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

When $\det(\epsilon_1^0 + t\epsilon_2^0)$ = 0 has at least two roots and $\epsilon(t) = \epsilon_1^0 + t\epsilon_2^0$ is never positive or negative definite

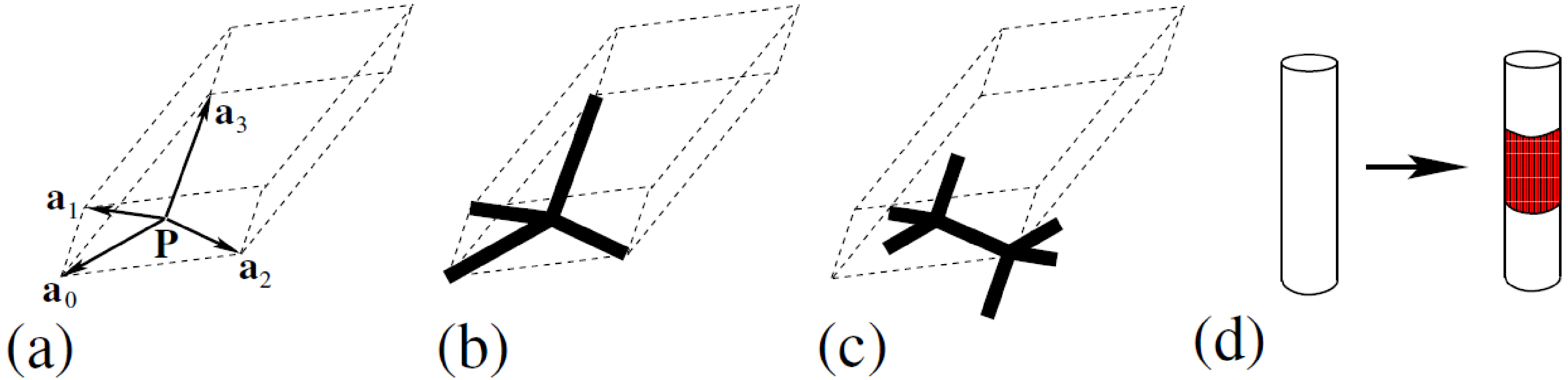
$$W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \epsilon_1^0, \epsilon_2^0) = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

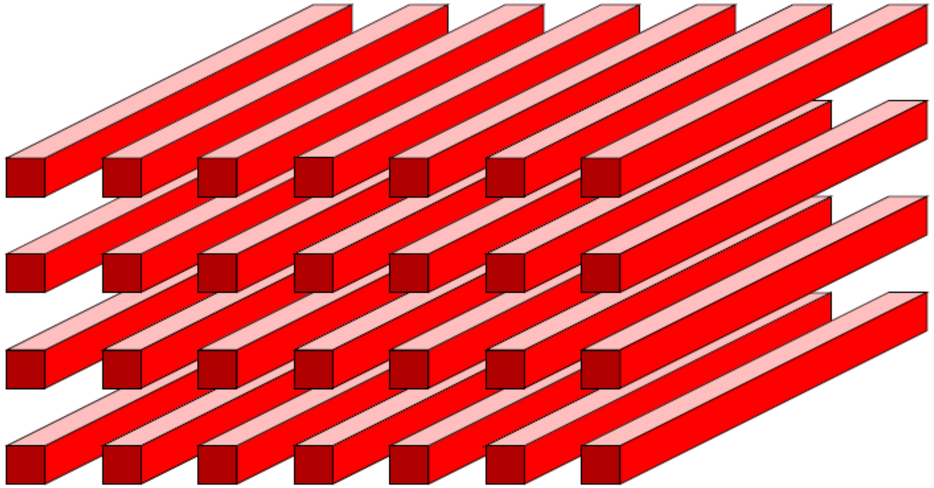
Idea of proof: Insert into the Avellaneda material a thin walled structure with sets of parallel walls:



Inside the walls put the appropriate multimode material

Modifying the pentamodes:





An exciting frontier in homogenization theory: space time microstructures

Systematically studied by Lurie, among others, but much remains to be done

To illustrate some novel features I will talk about a special class of space time microstructures supporting a new class of wave: field patterns

Joint work
With Ornella Mattei

Formulation of the problem

- Generic wave equation:

$$\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u(x,t)}{\partial x}\right)-\frac{\partial}{\partial t}\left(\beta(x,t)\frac{\partial u(x,t)}{\partial t}\right)=0$$

The coefficients are time – dependent $\rightarrow$ DYNAMIC MATERIALS

- Boundary conditions: The medium is infinite in the x–direction

- Initial conditions:

$$u(x,0)=g(x)$$
$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}=f(x)$$

[see, e.g., Lurie, 2007]

Realization of dynamic materials

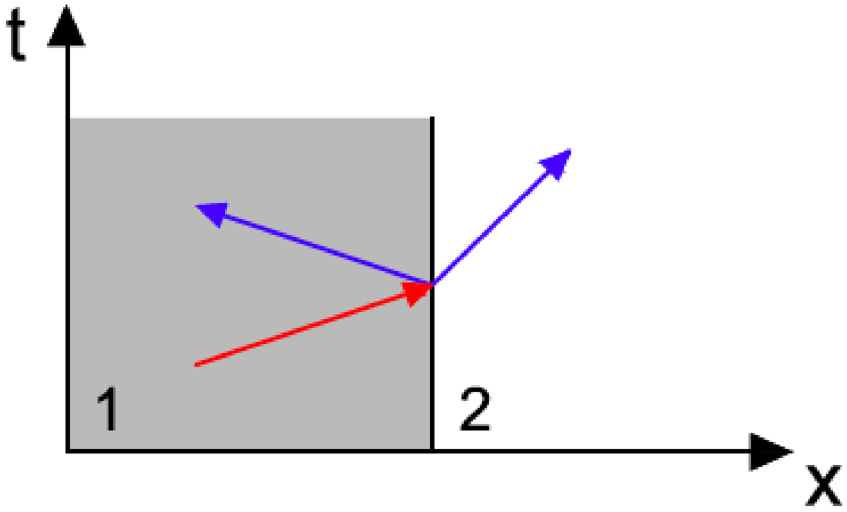
- Liquid crystals
- Ferroelectric, ferromagnetic materials
- Pump wave + small amplitudes waves [e.g. Louisell & Quate (1958)]
- Transmission line with modulated inductance [e.g. Cullen (1958)]
- Experiments and more references in [Honey & Jones (1958)]
- Dynamic modulation + photons [e.g. Fang et al. (2012), Boada et al. (2012), Celi et al. (2014), Yuan et al. (2016)]

...

- Walking droplets [e.g. Couder et al. (2005), Couder & Fort (2006), Bush (2015)]
- Time reversal [e.g. Fink (2016), Goussev et al. (2016)]

Dynamic composites

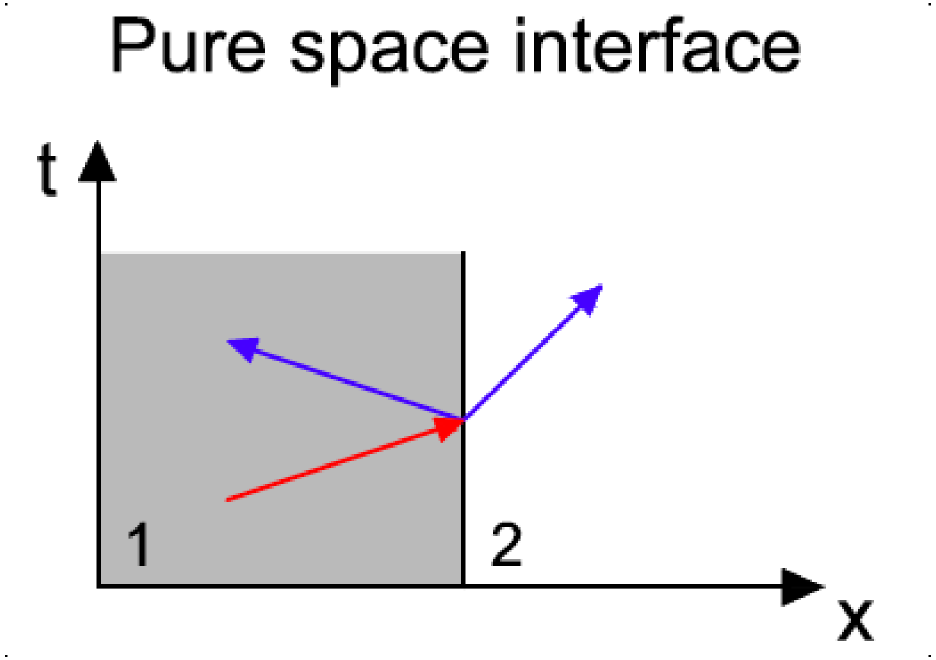
Pure space interface

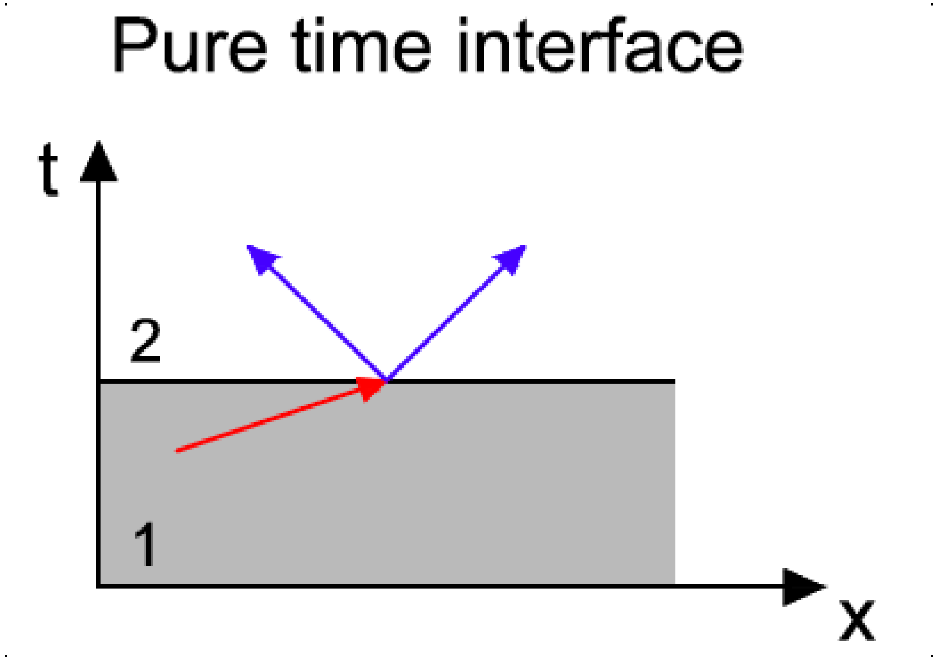


Pure time interface

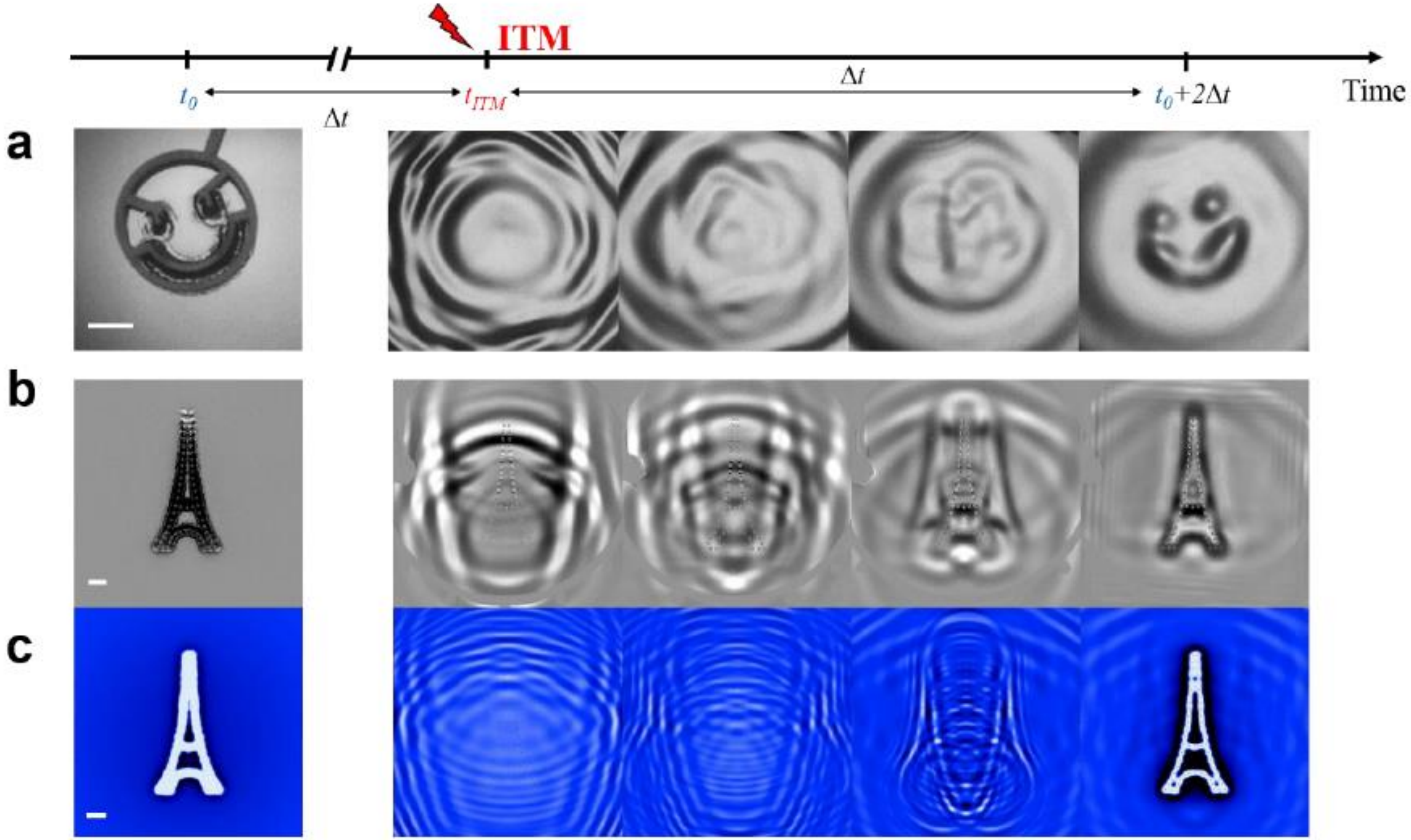
?

Dynamic composites





What happens at a time interface?



Bacot, Labousse, Eddi, Fink, and Fort, Nature 2016

Thinking of the wave equation as a conductivity problem

$$\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \text{where} \quad \nabla \cdot \mathbf{j} = 0, \quad \mathbf{e} = -\nabla V,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{pmatrix} \alpha(\mathbf{x}) & 0 \\ 0 & -\beta(\mathbf{x}) \end{pmatrix}, \quad \begin{matrix} \text{material } 1 & \rightarrow & \alpha_1, \beta_1 \\ \text{material } 2 & \rightarrow & \alpha_2, \beta_2 \end{matrix}$$

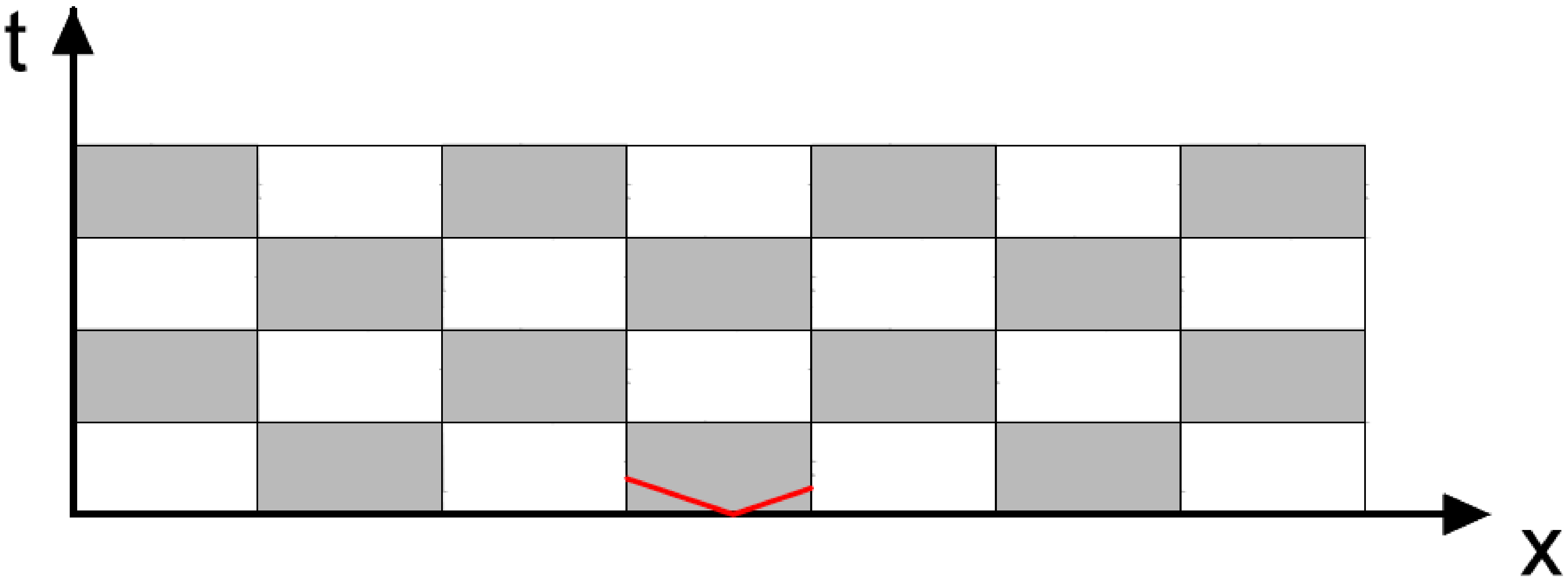
$$\frac{\partial}{\partial x_1}\left(\alpha(x_1, x_2)\frac{\partial V(x_1, x_2)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\beta(x_1, x_2)\frac{\partial V(x_1, x_2)}{\partial x_2}\right) = 0$$

N.B. Hyperbolic materials!! [See, e.g. the review Poddubny, Iorsh, Belov, Kivshar, 2013]

$$\alpha_i \frac{\partial^2 V_i}{\partial x^2} - \beta_i \frac{\partial^2 V_i}{\partial t^2} = 0, \qquad i = 1, 2$$

D'Alembert solution : $V_i(x, t) = V_i^+(x - c_i t) + V_i^-(x + c_i t)$ $\qquad c_i = \sqrt{\dfrac{\alpha_i}{\beta_i}}$

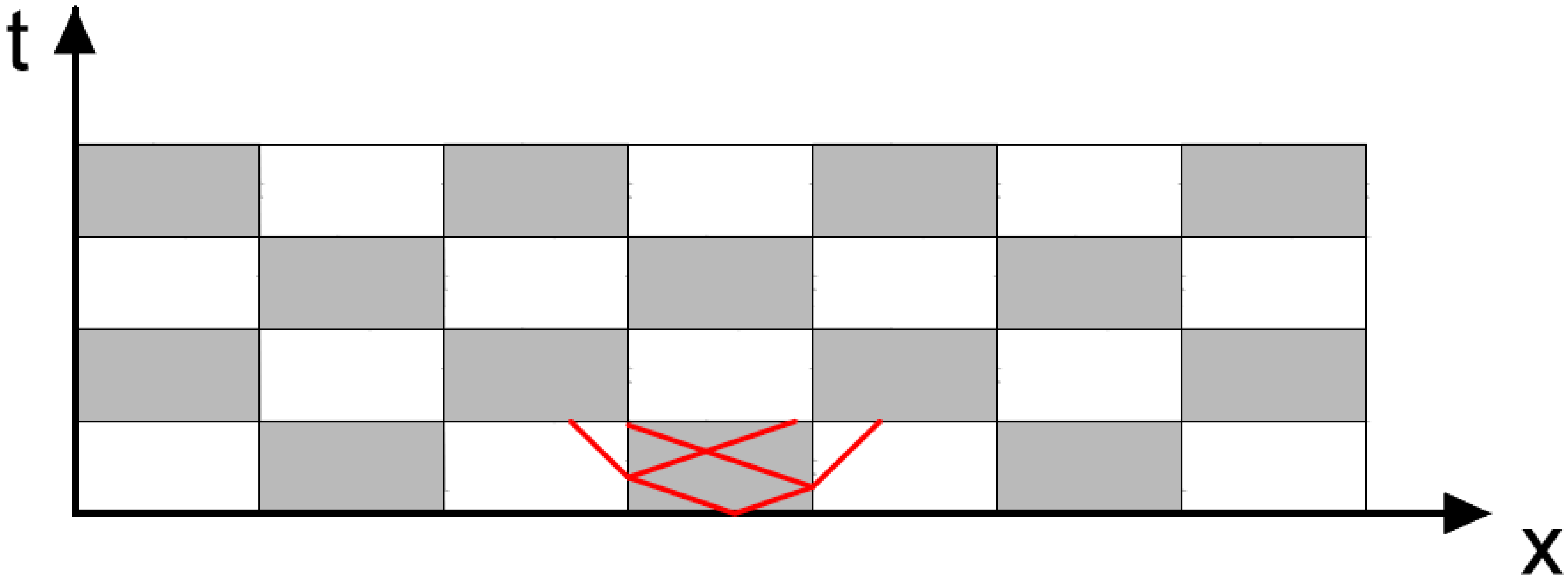
Evolution of a disturbance in a space-time checkerboard



Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla V_1 = \mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla V_2 \end{cases}$$

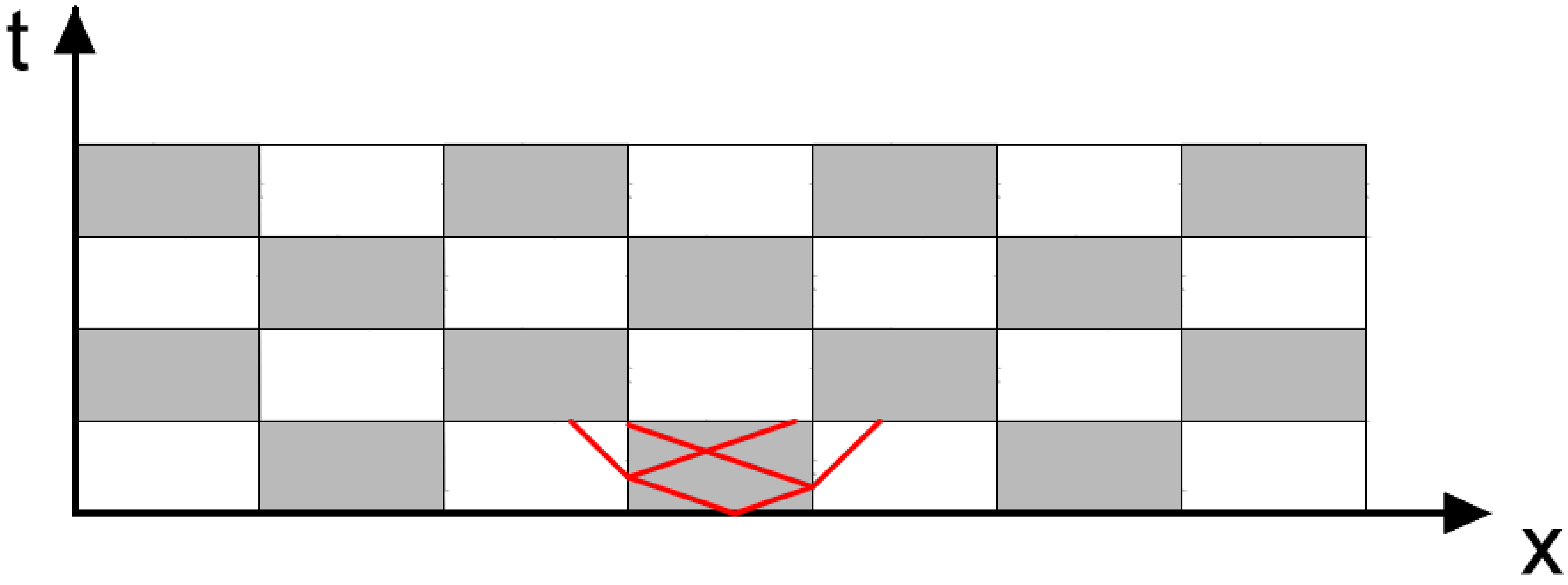
Evolution of a disturbance in a space-time checkerboard



Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \sigma_1 \nabla V_1 = \mathbf{n} \cdot \sigma_2 \nabla V_2 \end{cases}$$

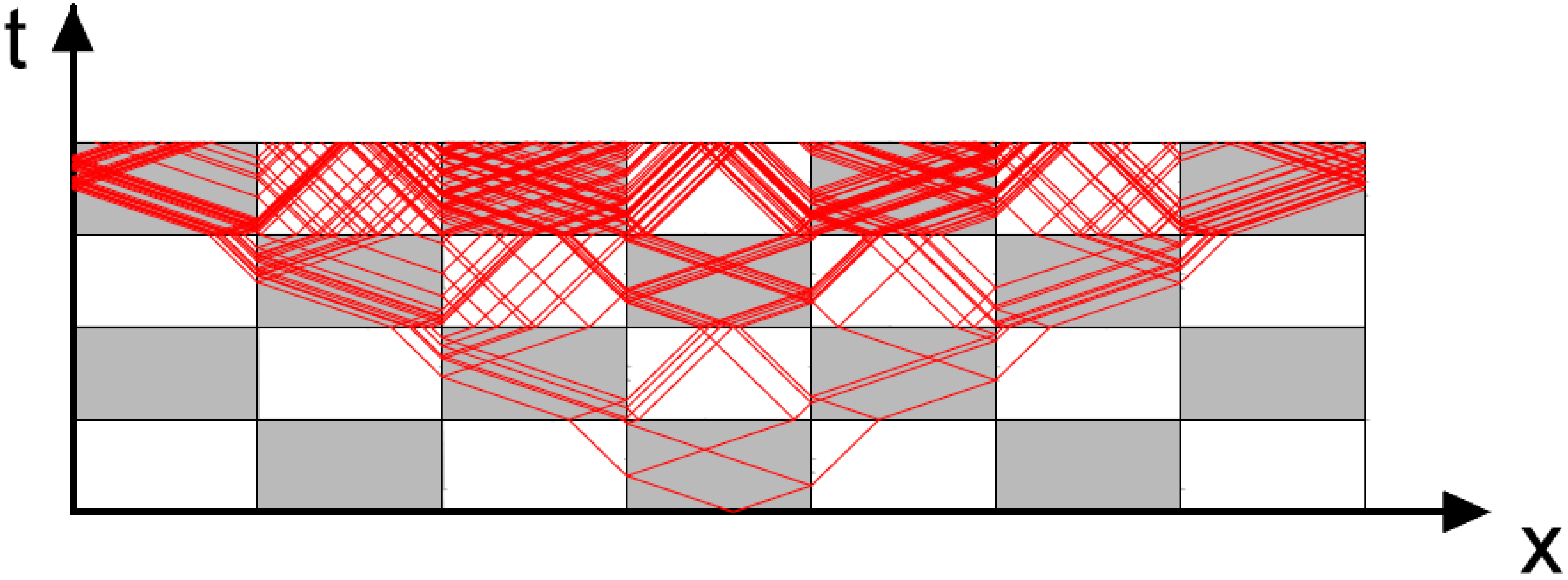
Evolution of a disturbance in a space-time checkerboard



Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \sigma_1 \nabla V_1 = \mathbf{n} \cdot \sigma_2 \nabla V_2 \end{cases}$$

Evolution of a disturbance in a space-time checkerboard

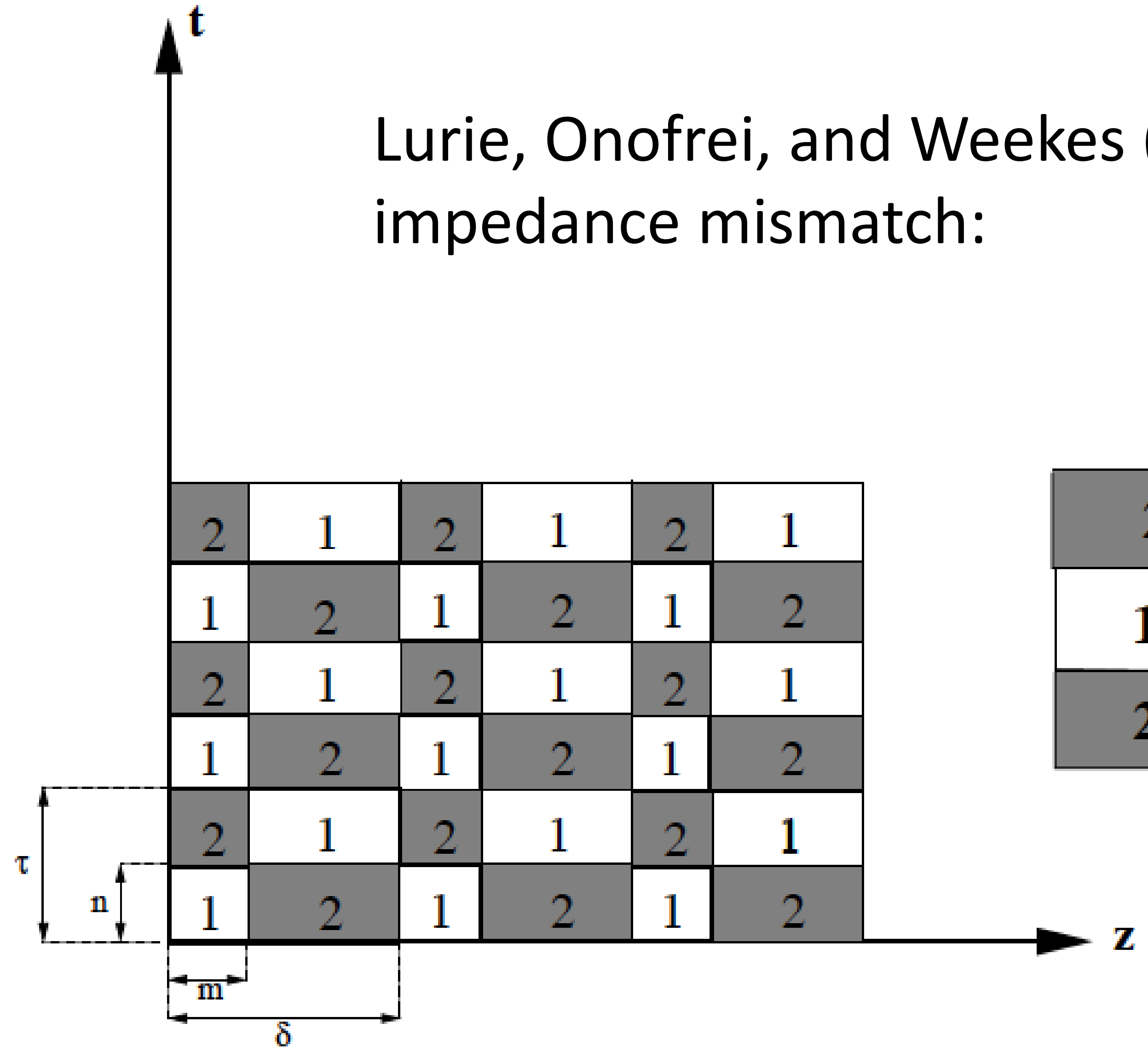


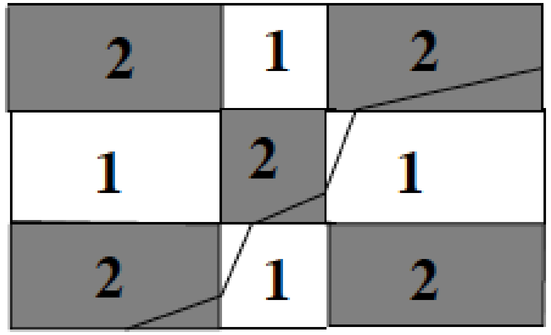
Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla V_1 = \mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla V_2 \end{cases}$$

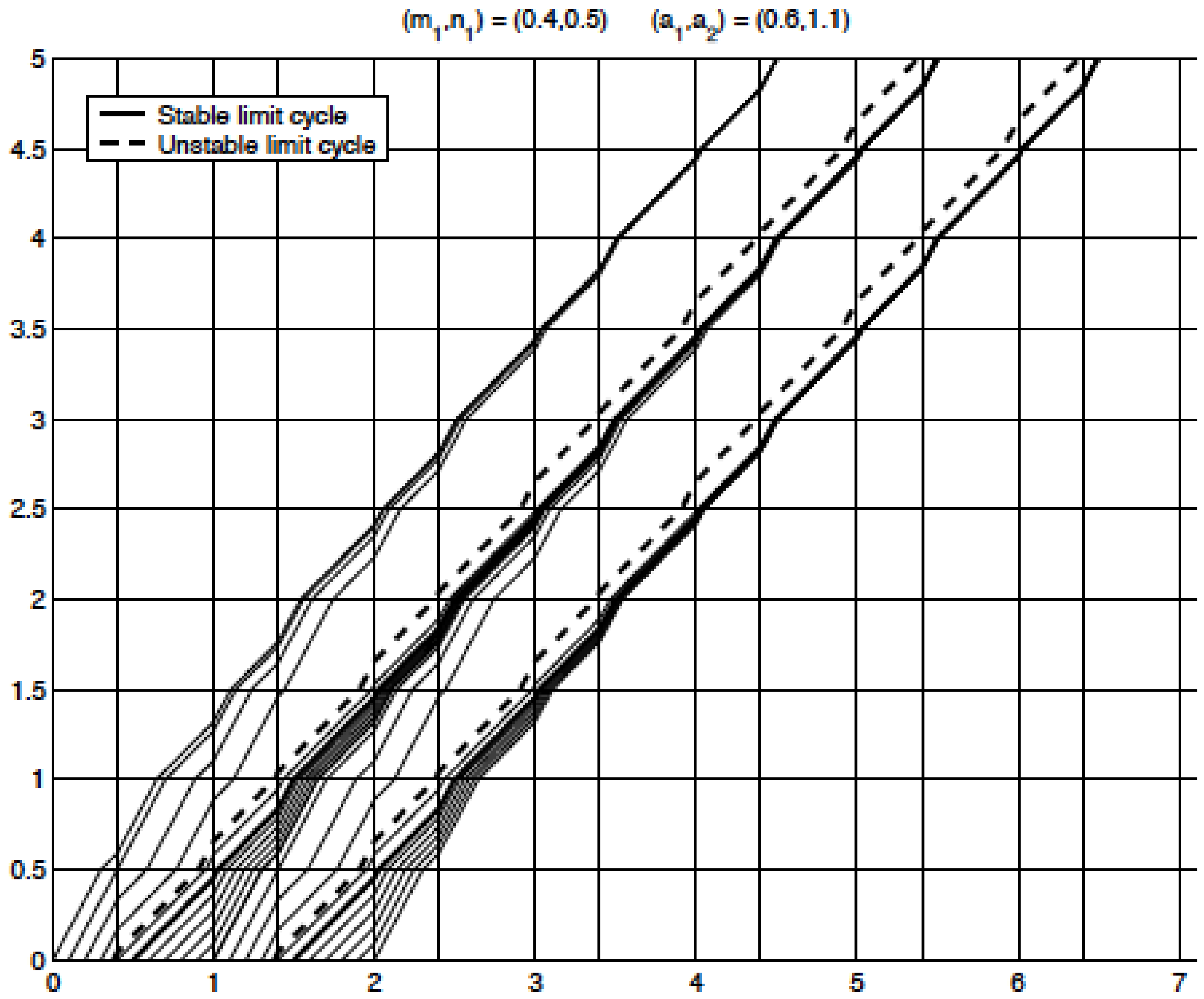
How to avoid this complicated cascade?

Lurie, Onofrei, and Weekes (2009) suggested having a zero impedance mismatch:





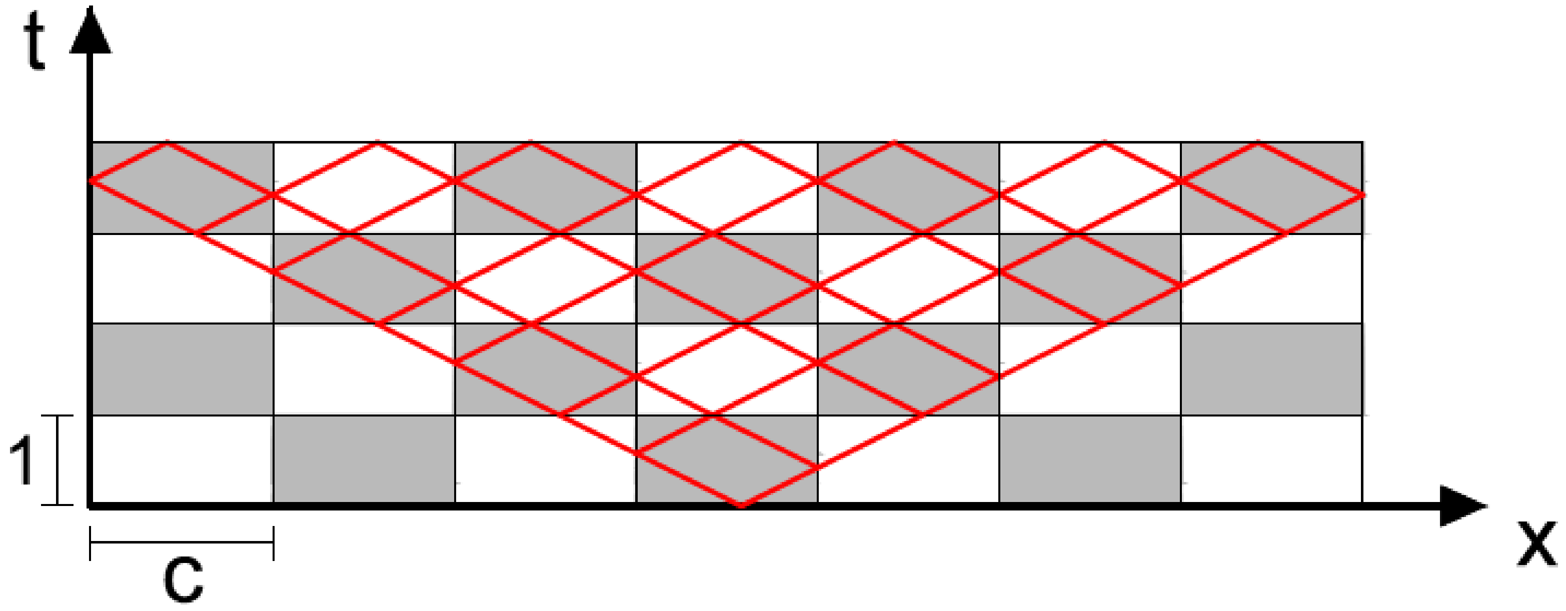
Curiously they found accumulations of the characteristic lines:



A bit like a shock but in a linear medium!

Field patterns in a space-time checkerboard

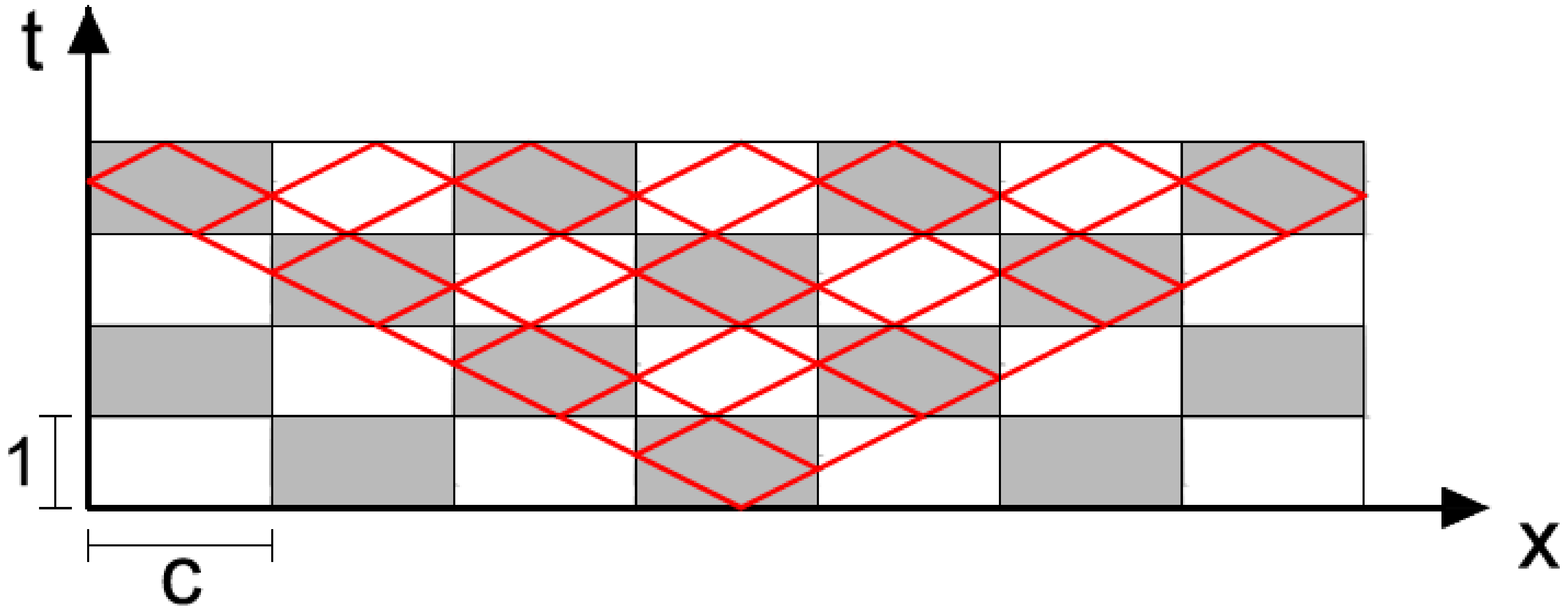
$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



Field patterns are a new type of wave propagating along orderly patterns of characteristic lines!!!

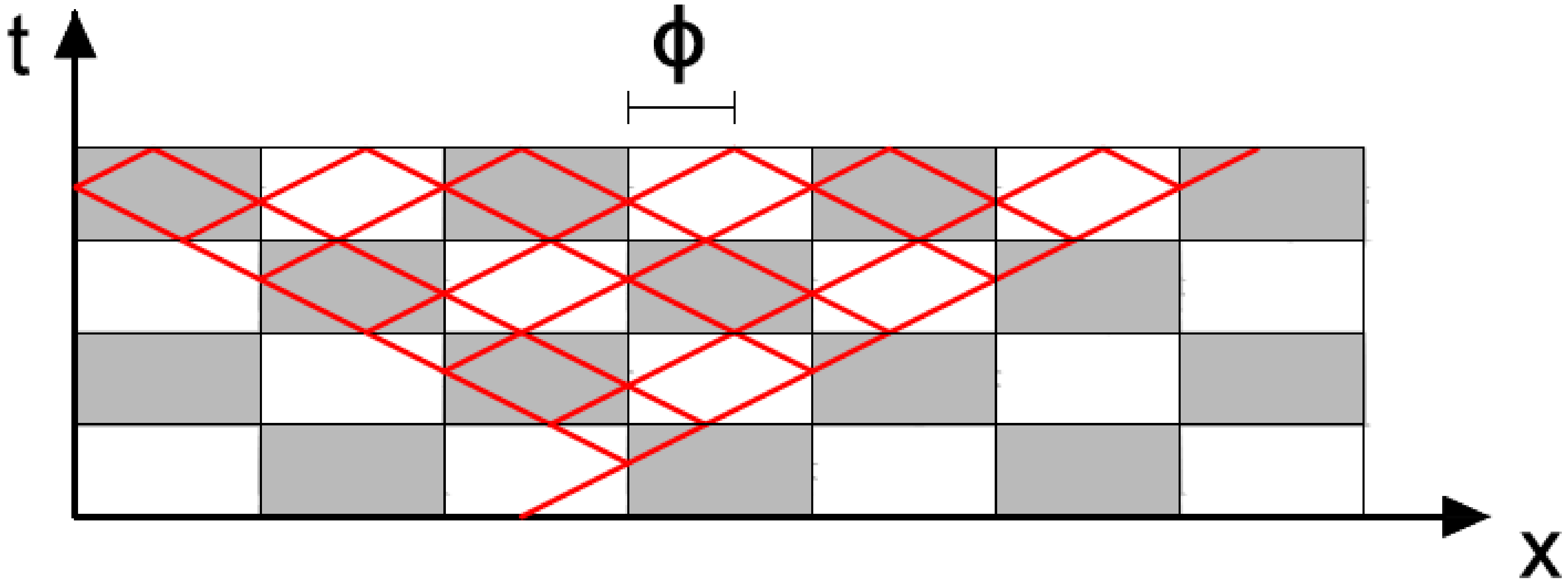
Field patterns in a space-time checkerboard

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



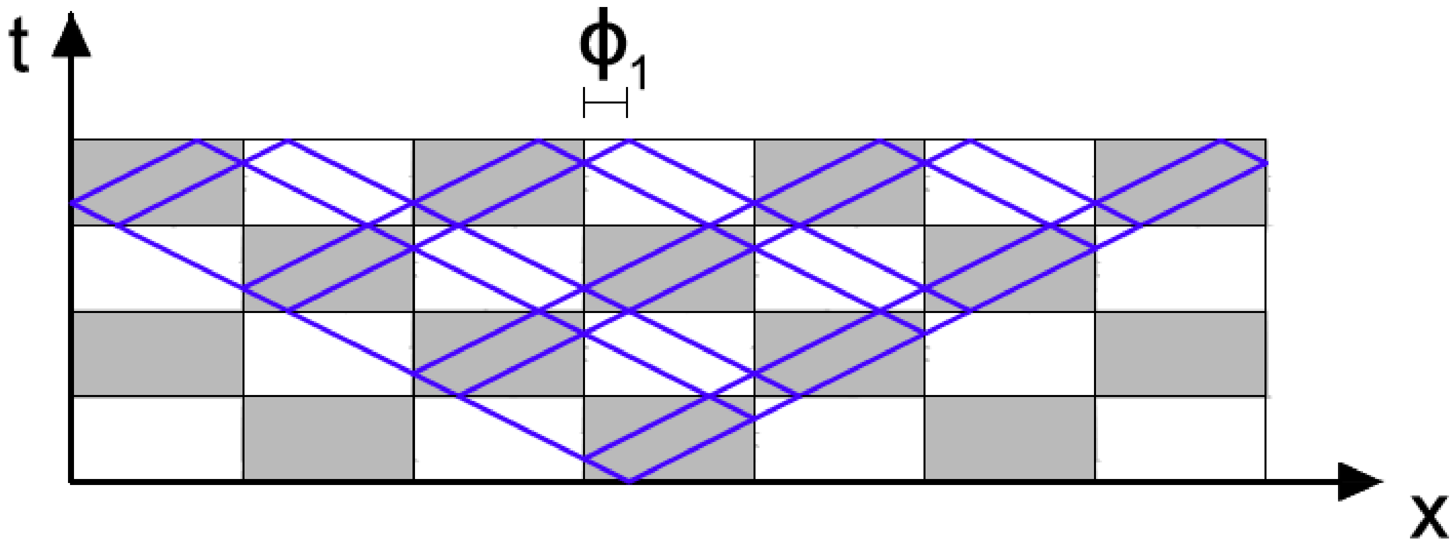
Families of field patterns

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



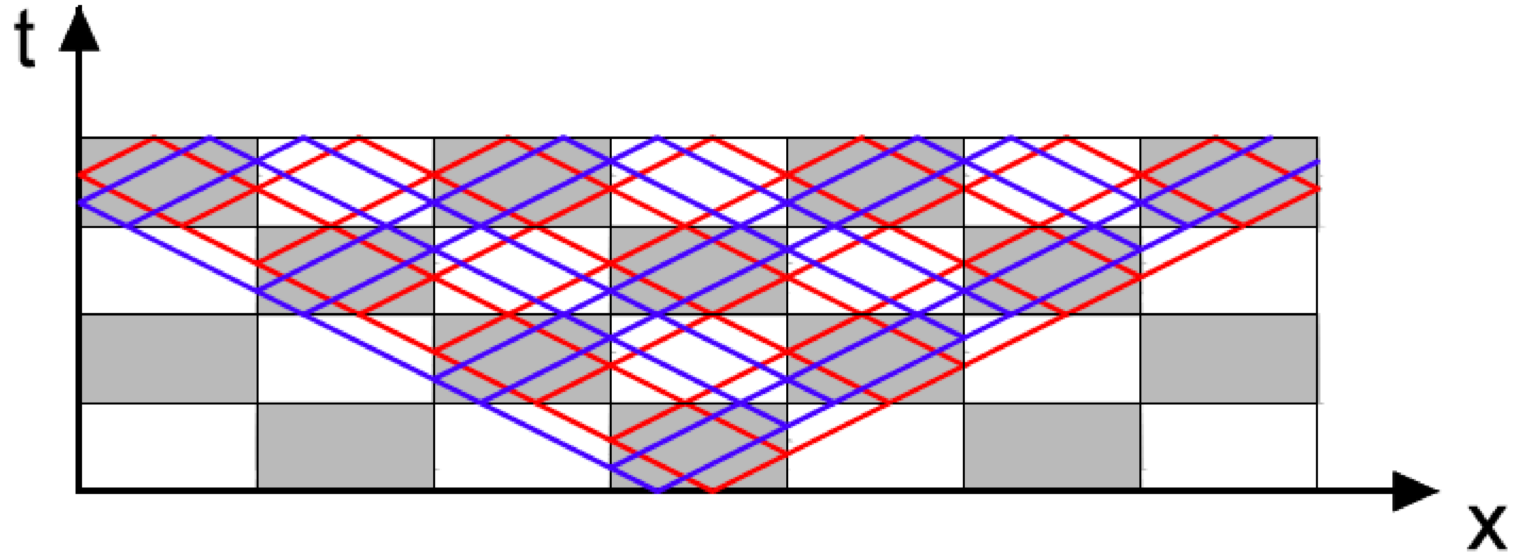
Families of field patterns

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



Field patterns are a new type of wave propagating along orderly patterns of characteristic lines which arise in specific space-time microstructures whose geometry in one spatial dimension plus time is somehow commensurate to the slope of the characteristic lines.

Multidimensional nature of field patterns

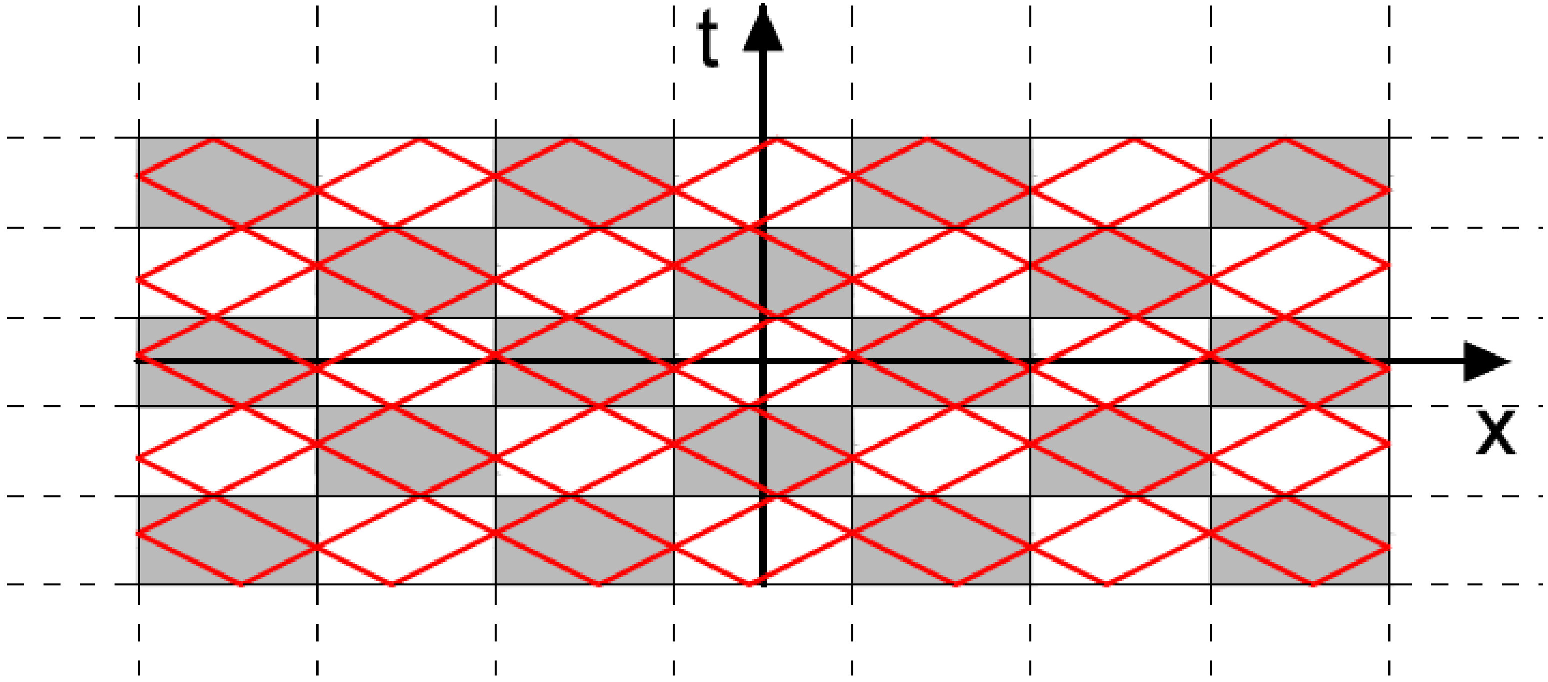


$$V(x,t) = \sum_{i=1}^{m} V_{\phi_i}(x,t)$$

Multidimensional space: $V(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} V_{\phi_i}(x_i, t)$

Multicomponent potential: $\mathbf{V}(x,t)$

PT–symmetry of field patterns

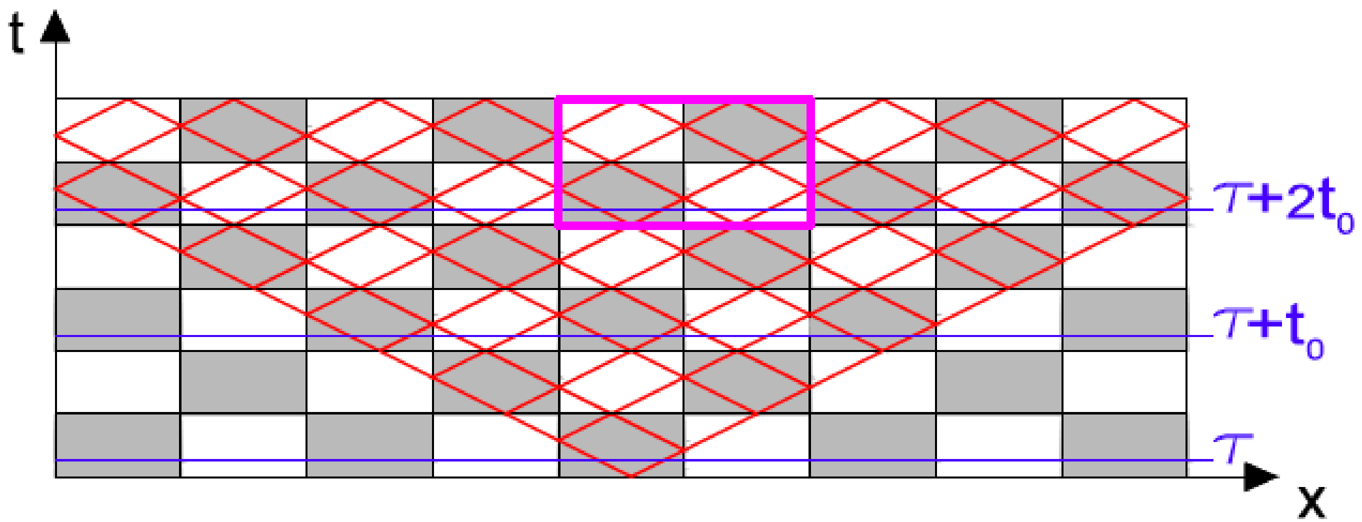


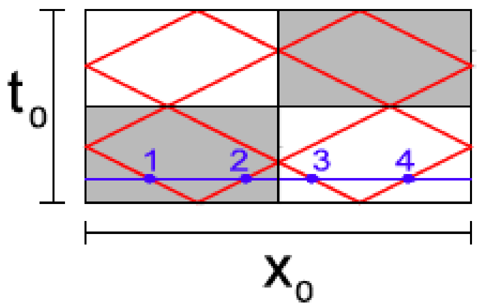
[Quantum physics, e.g., Bender and Boettcher, 1998,
Optics, e.g., Zyablovsky et al., 2014]

Unbroken PT–symmetry → real eigenvalues

Broken PT–symmetry → complex conjugate eigenvalues

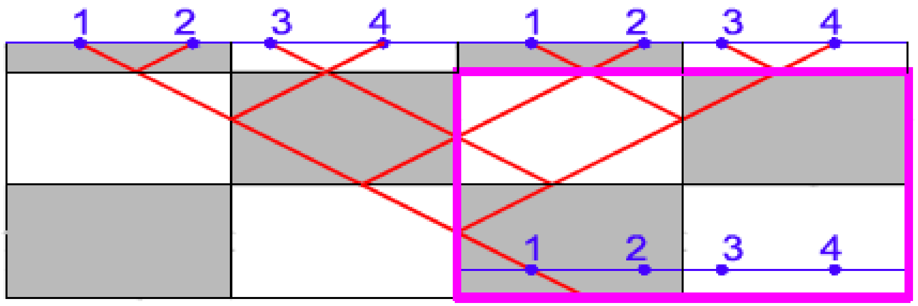
The transfer matrix





$$j(k,m,n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k',m',n)$$

$$T_{(k,m),(k,'m')} = G_{k,k'}(m-m')$$

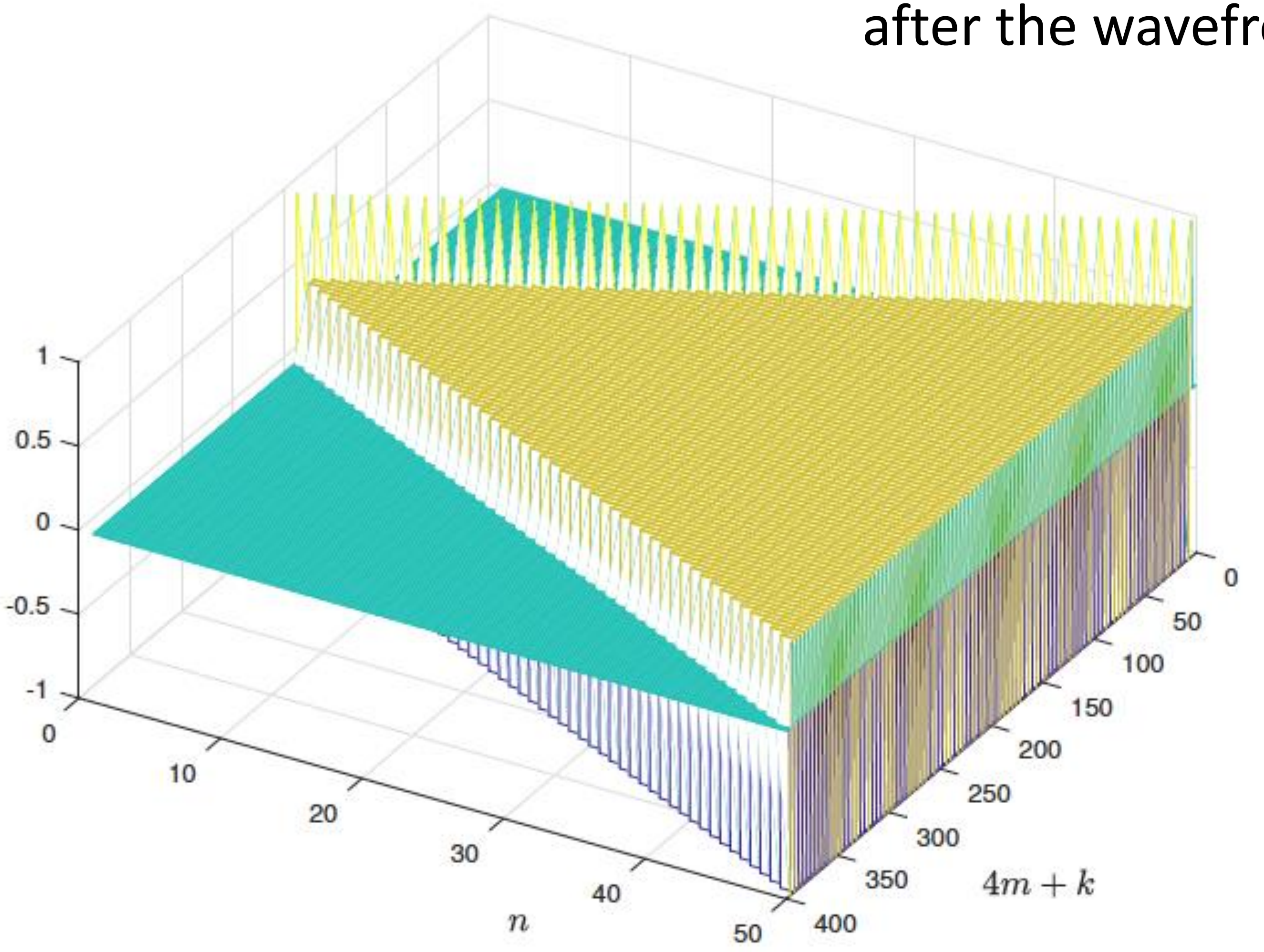


T depends only on γ_i

T is PT–symmetric

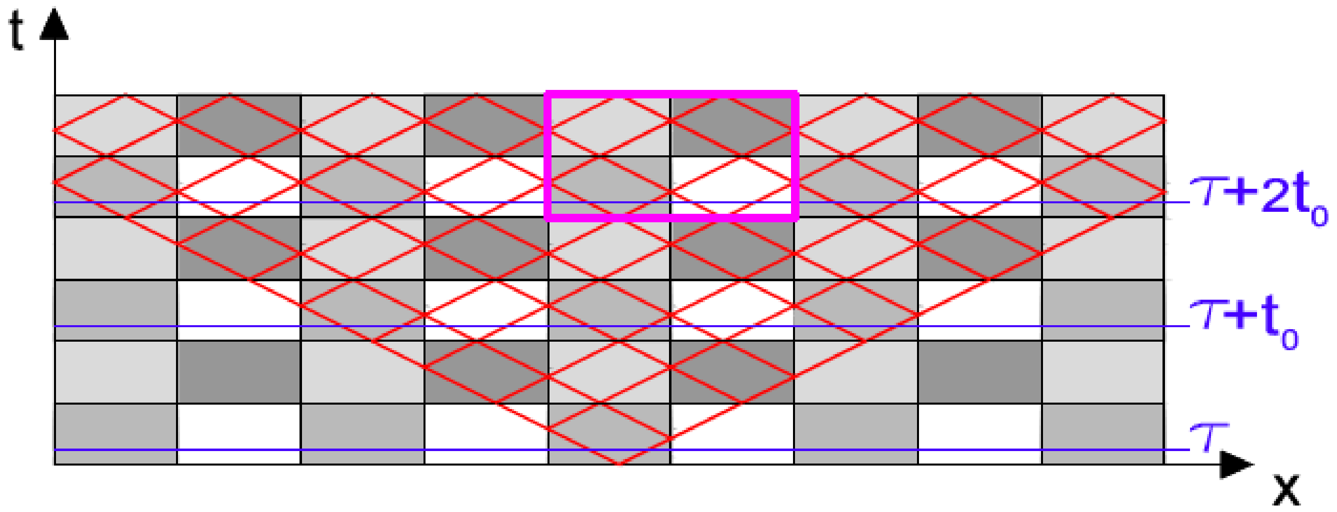
Evolution in time of the current distribution

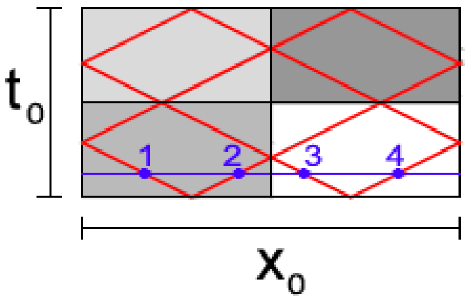
Note: oscillations continue after the wavefront!



Four–component space–time checkerboard

$$c_1 = c_2 = c_3 = c_4$$

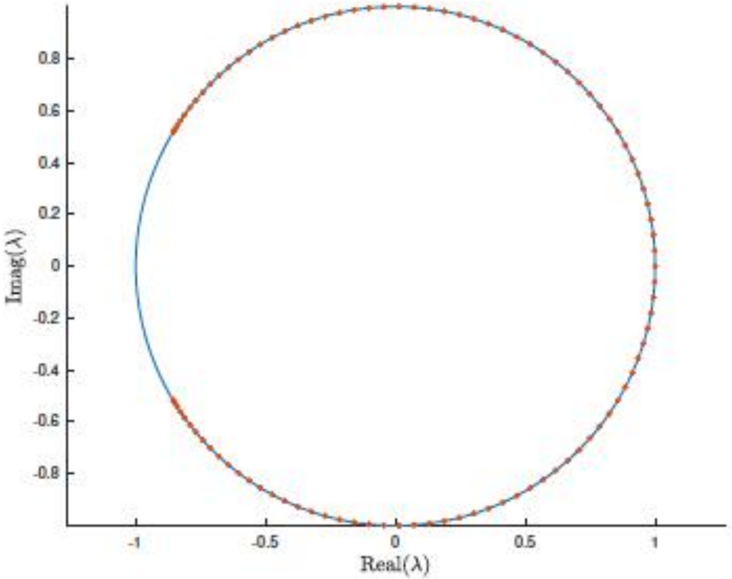


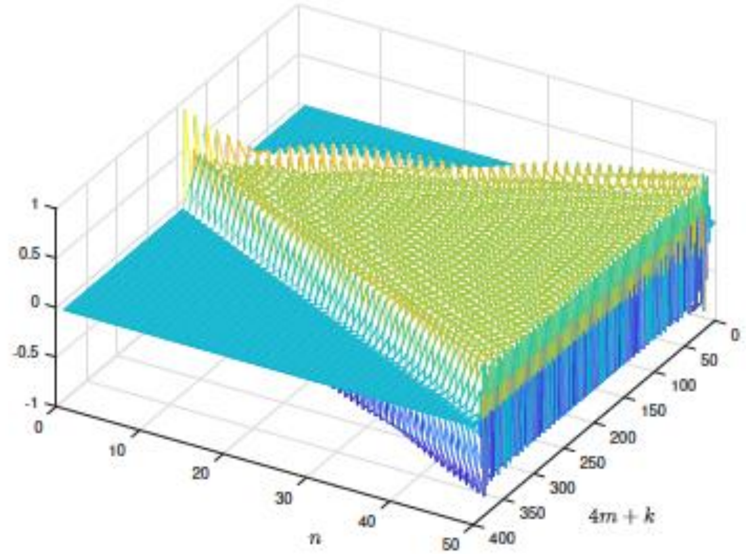


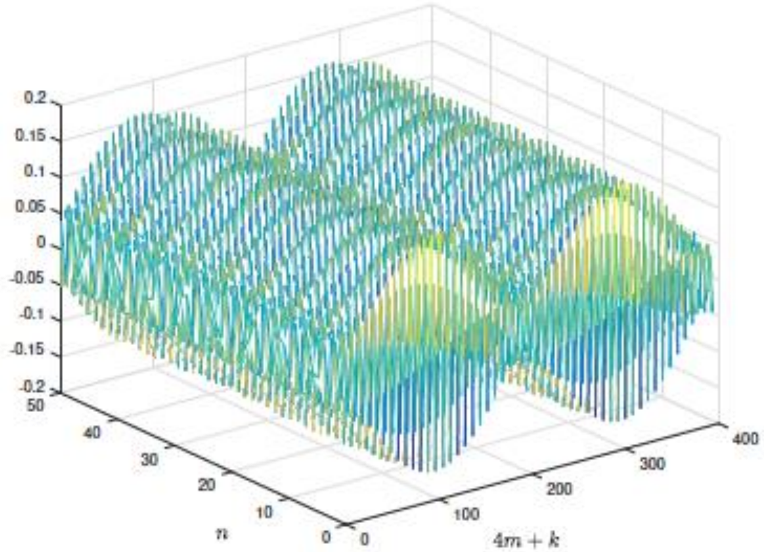
For some combinations of γ_1, γ_2, γ_3, γ_4 : UNBROKEN PT–symmetry

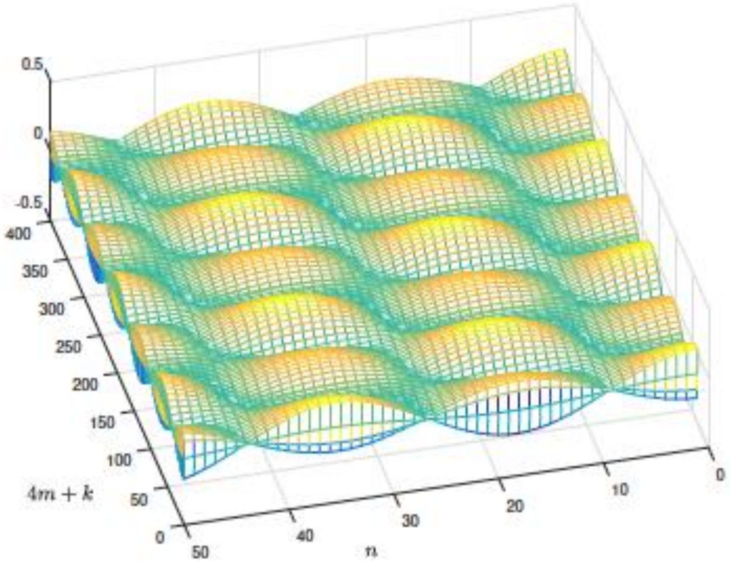
For other combinations of γ_1, γ_2, γ_3, γ_4 : BROKEN PT–symmetry

Unbroken PT–symmetry for the four–phase checkerboard

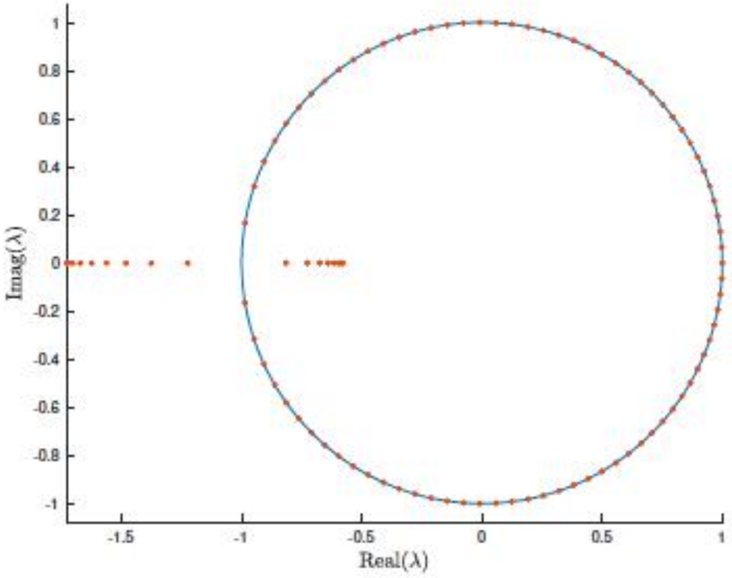


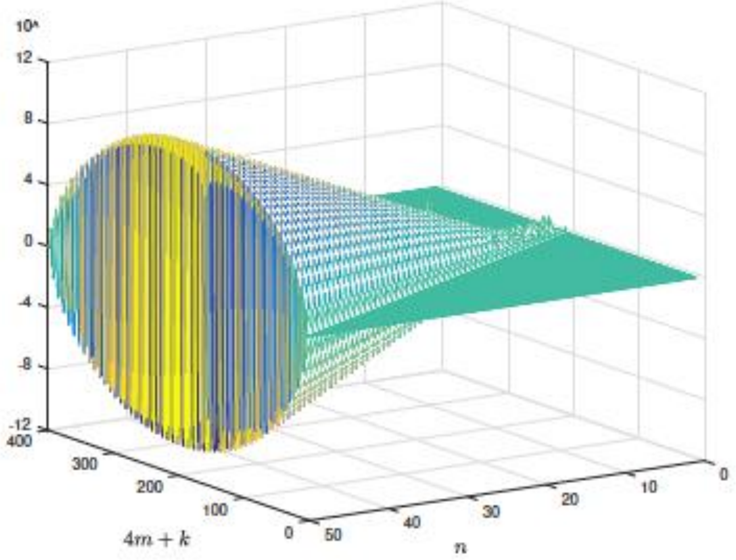






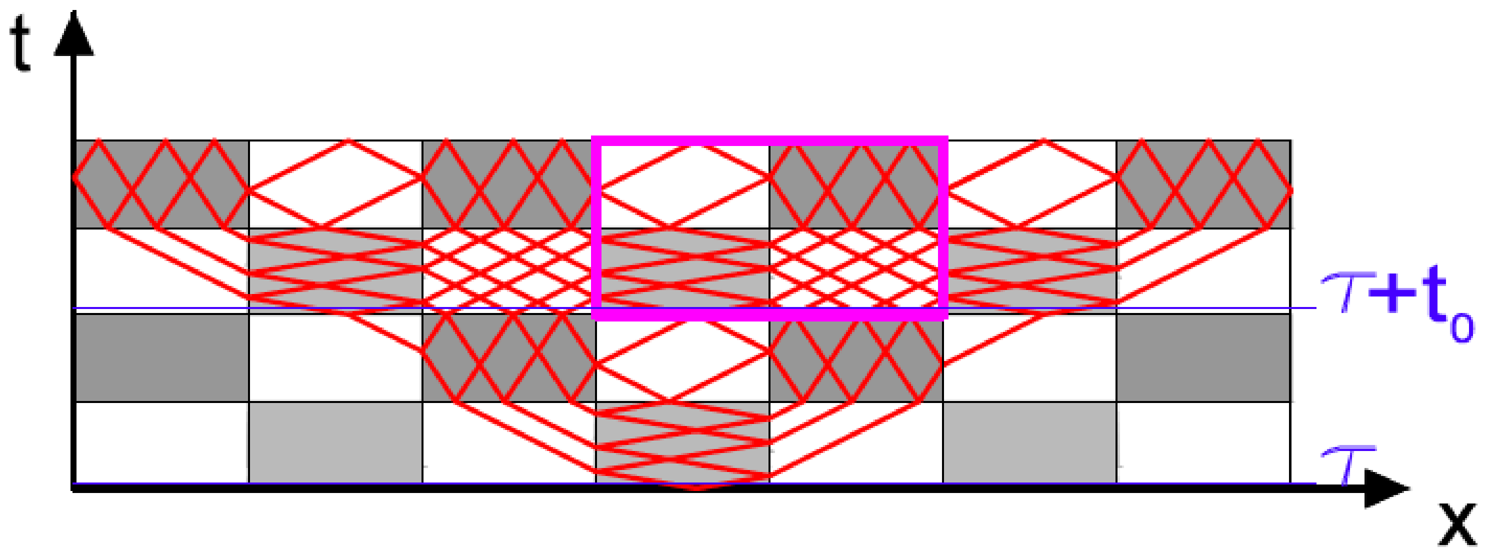
Broken PT–symmetry for the four–phase checkerboard





Three–phase space–time checkerboard

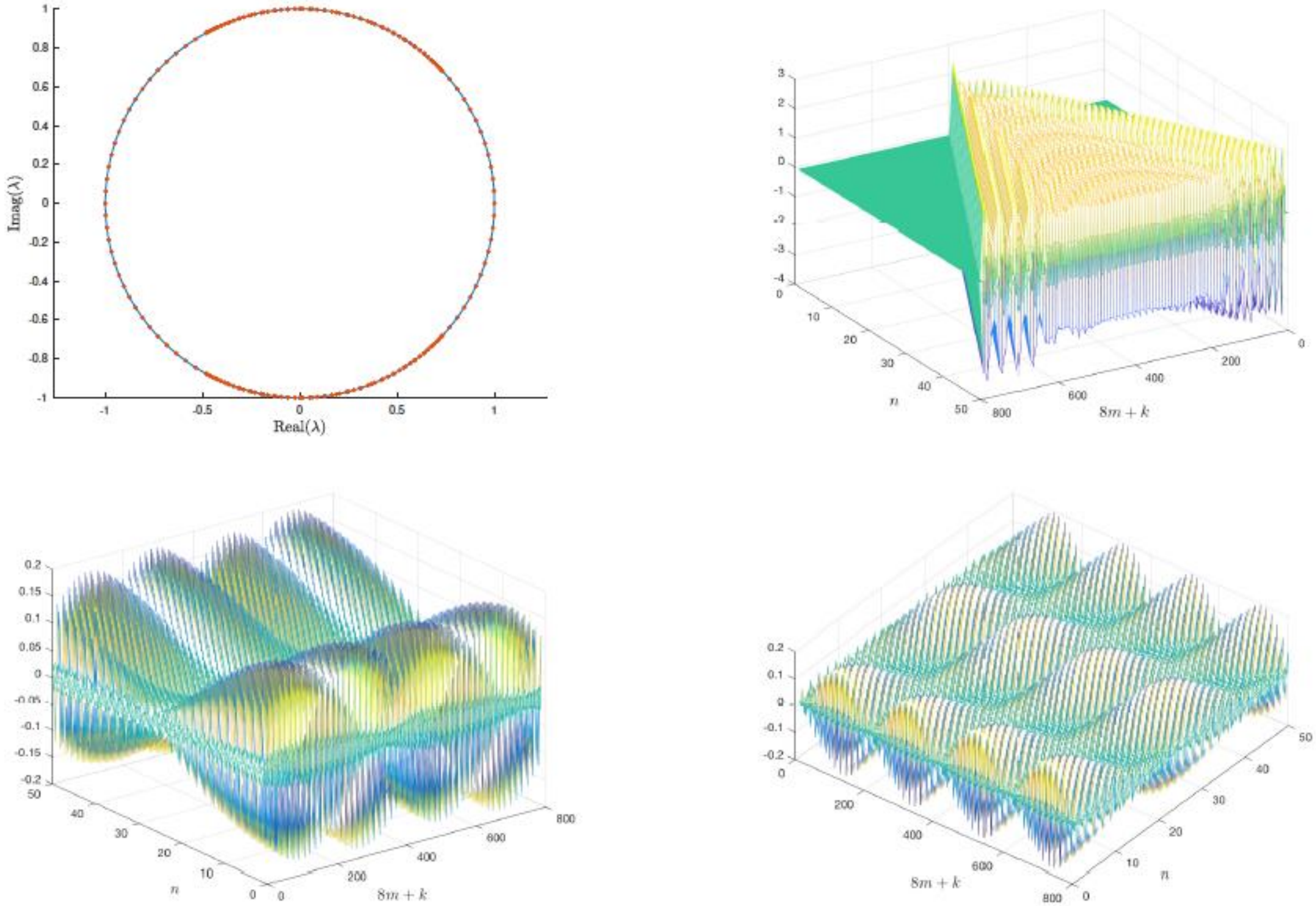
$$c_2/c_1 = c_1/c_3 = 3$$



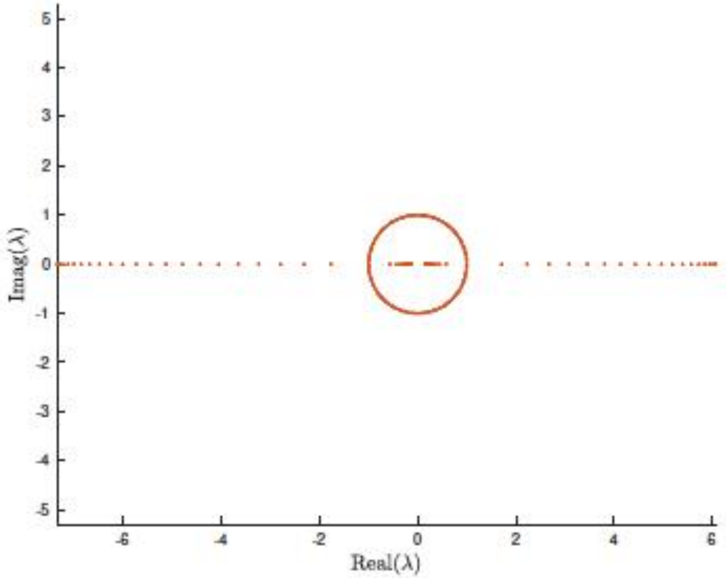
For some combinations of γ_1, γ_2, γ_3 : UNBROKEN PT–symmetry

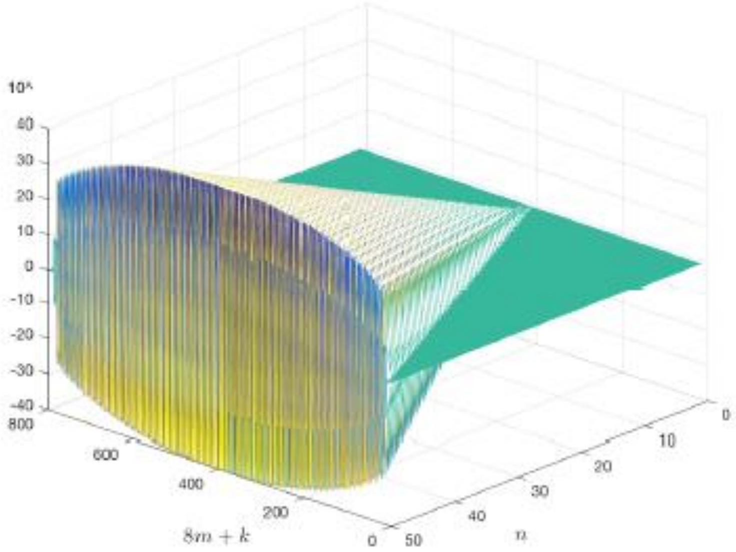
For other combinations of γ_1, γ_2, γ_3 : BROKEN PT–symmetry

Unbroken PT–symmetry for the three–phase checkerboard



Broken PT–symmetry for the three–phase checkerboard

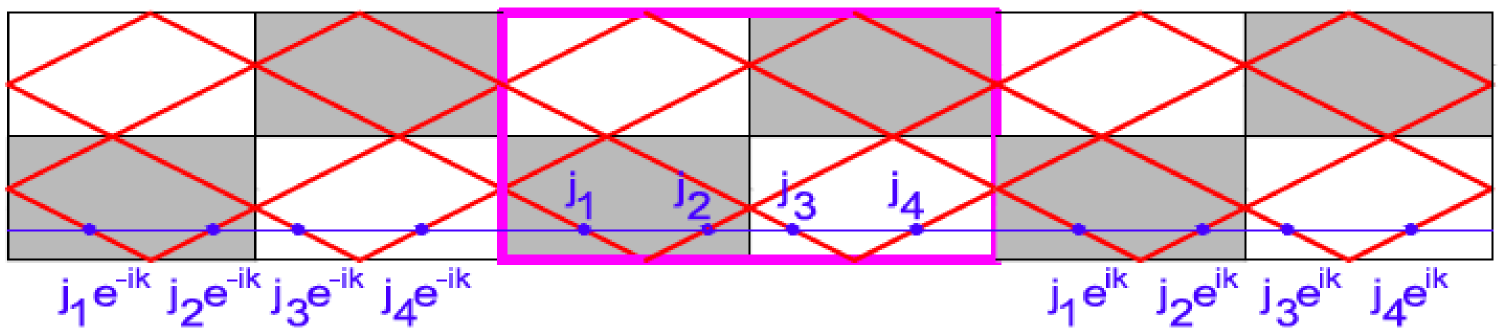




Bloch–Floquet theory applied to field patterns

Periodicity with respect to x:

$$j(l, m+s, n) = \exp(\mathrm{i}ks)\, j(l, m, n)$$



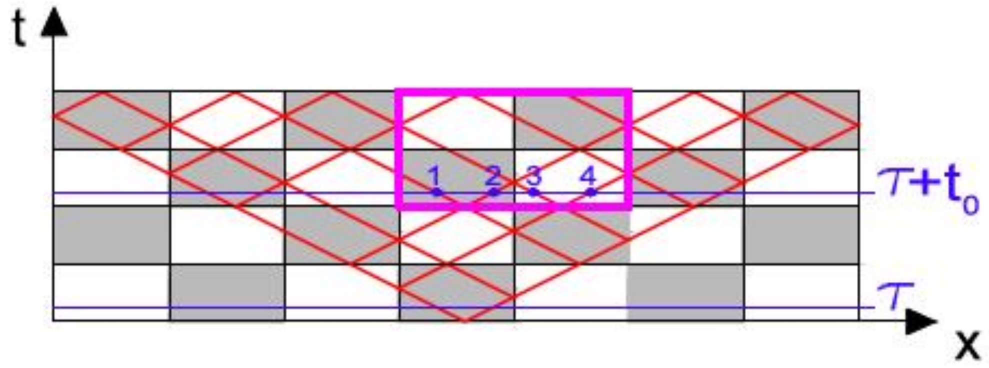
Periodicity with respect to t:

$$j(l, m, n+q) = \exp(\mathrm{i}\,\omega\, q) j(l, m, n)$$

Recall: $j(l, m, n+q) = \lambda^q j(l, m, n)$, then

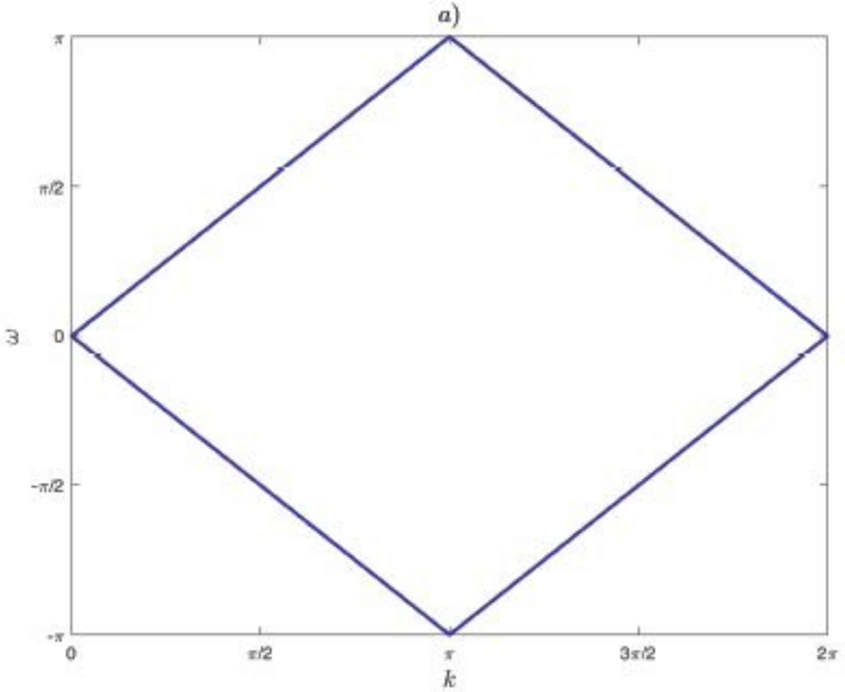
Dispersion relation : $\lambda(k) = \exp(\mathrm{i}\,\omega)$

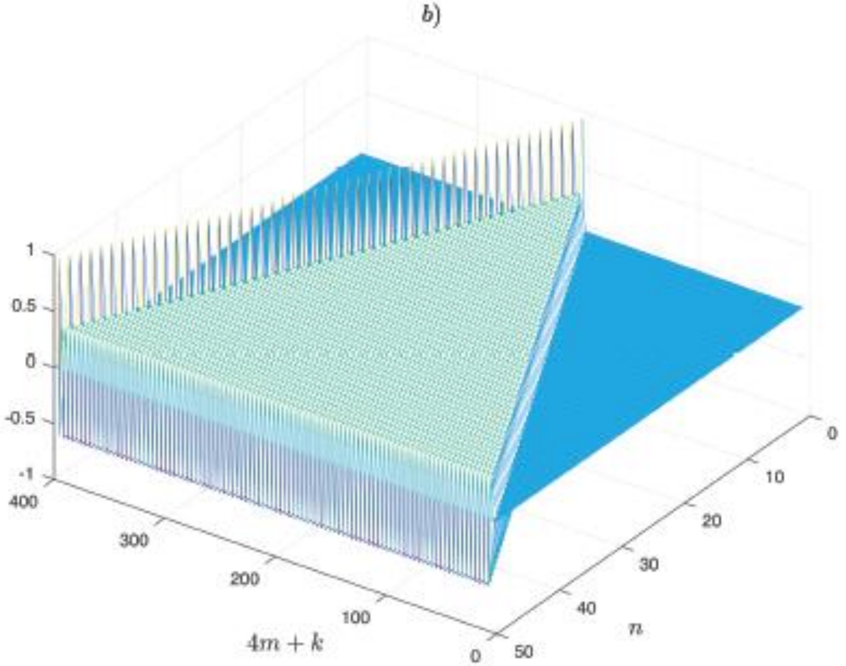
Dispersion diagram for the two–phase checkerboard



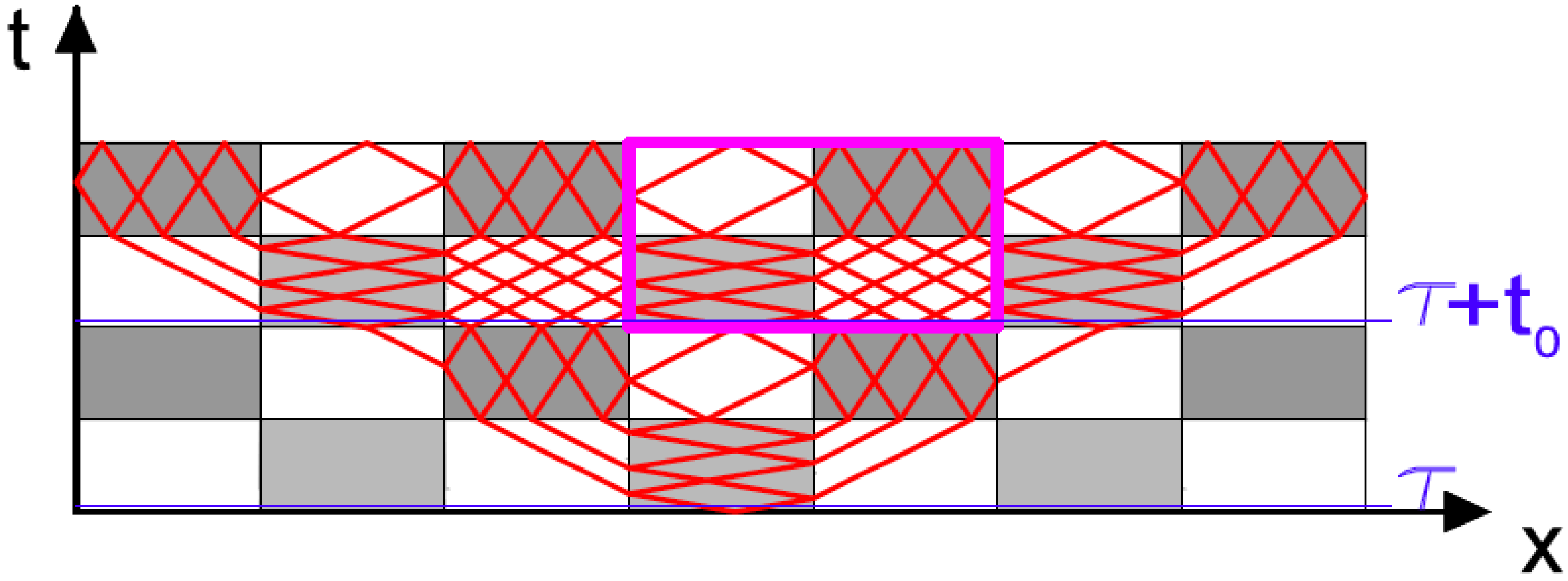
$$\lambda(k) = \exp(\pm \mathrm{i} k)$$

Dispersion relation: $\omega = \pm k$

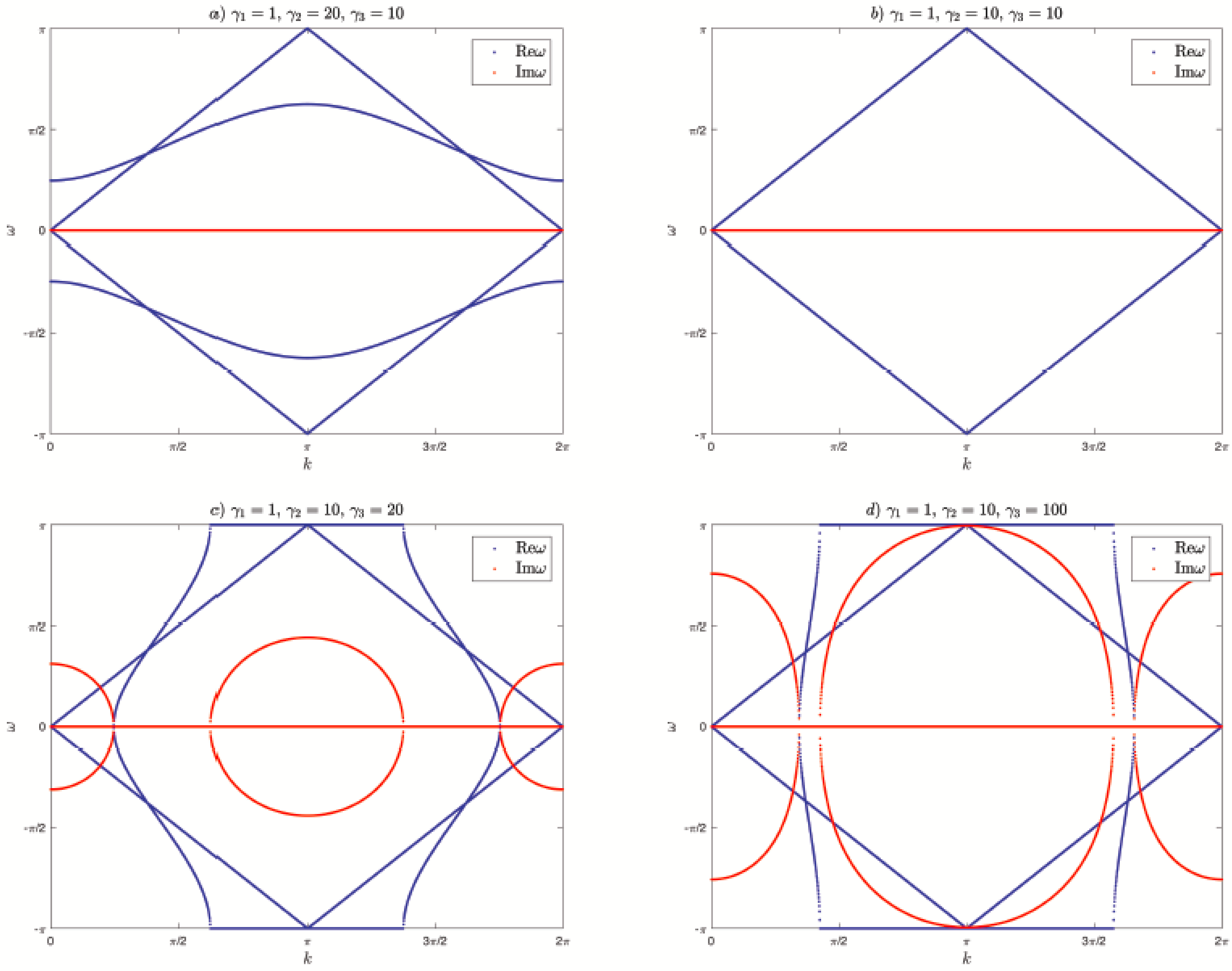




Three–phase checkerboard with phases having speed in a certain ratio



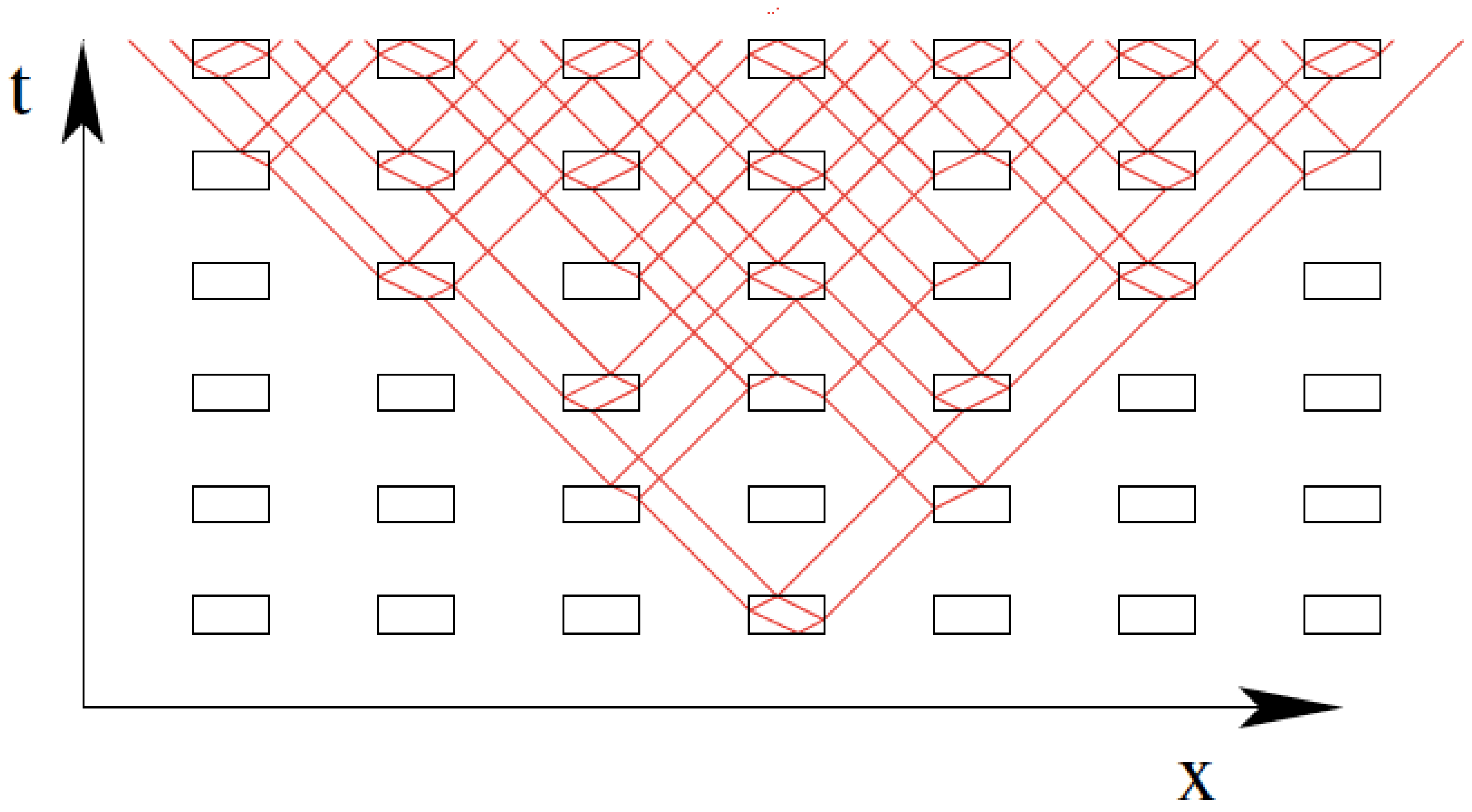
Dispersion diagrams for the three–phase checkerboard



Bloch Waves are:

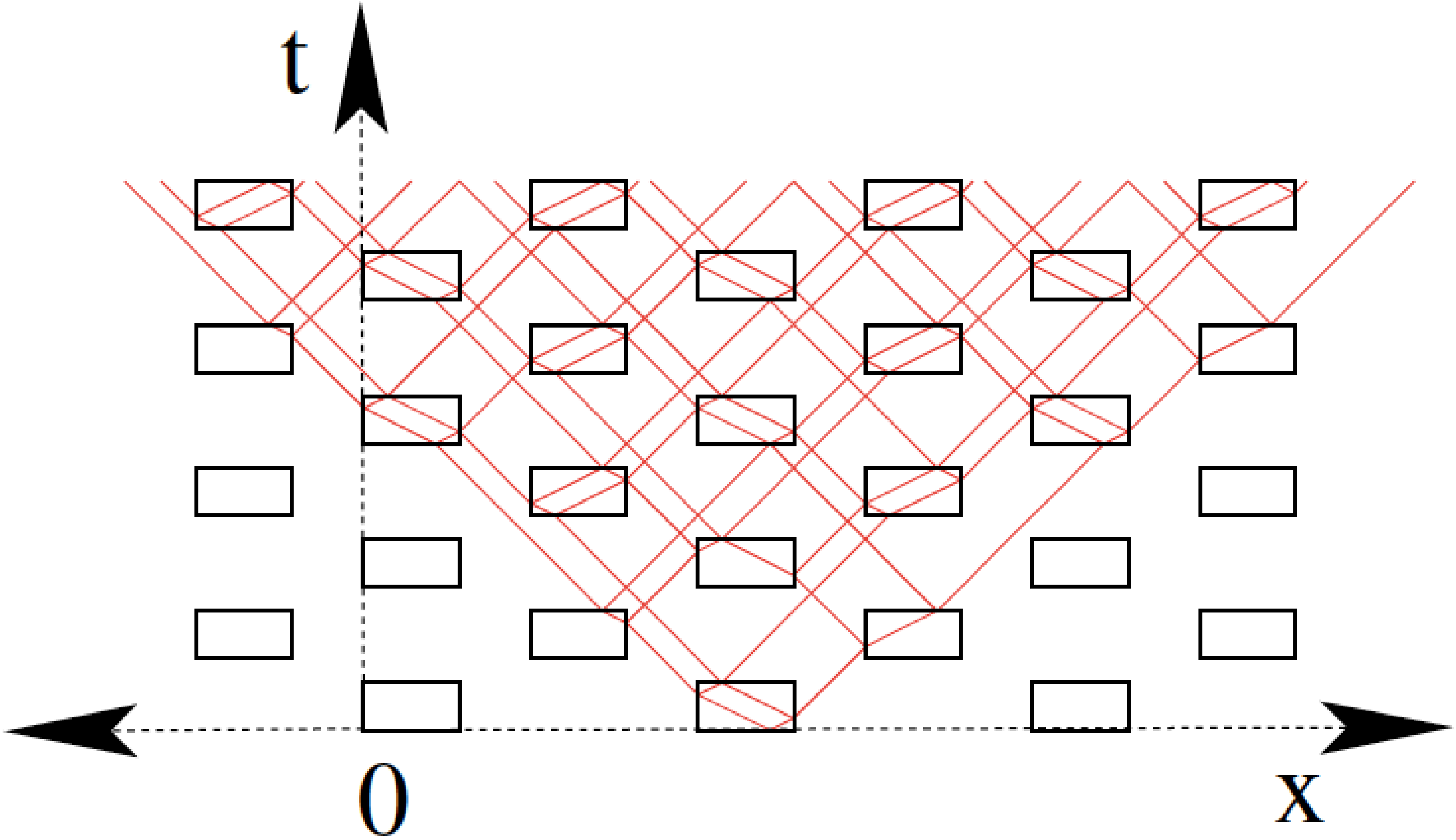
Infinitely Degenerate!

What about other field pattern geometries?



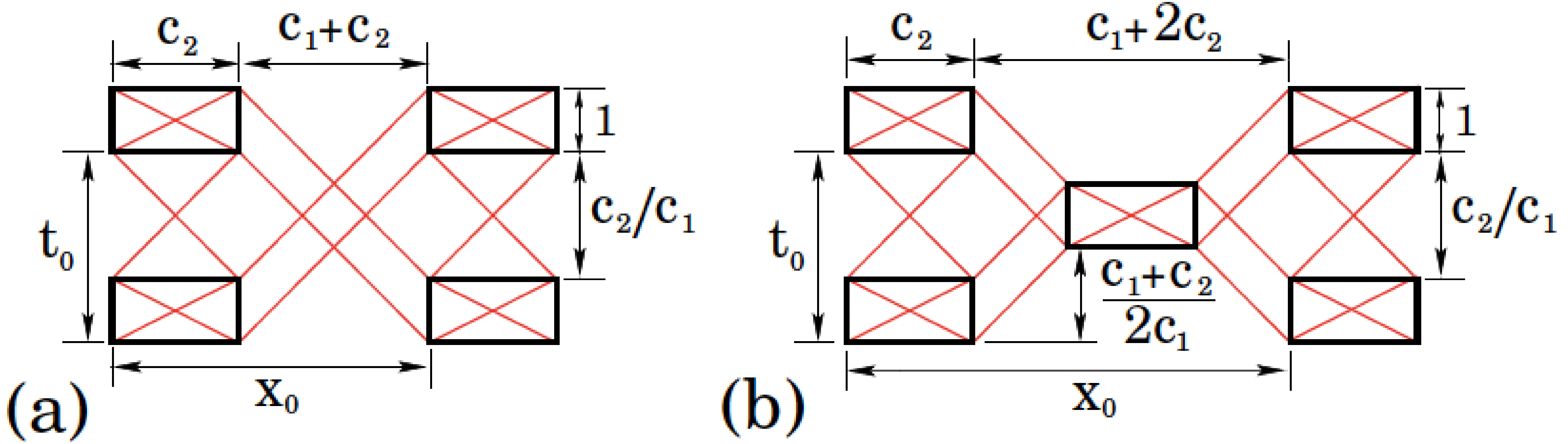
Characteristic lines lie on a pattern–the field pattern.
Note the P-T symmetry of the microstructure.

Alternatively one can have staggered inclusions:

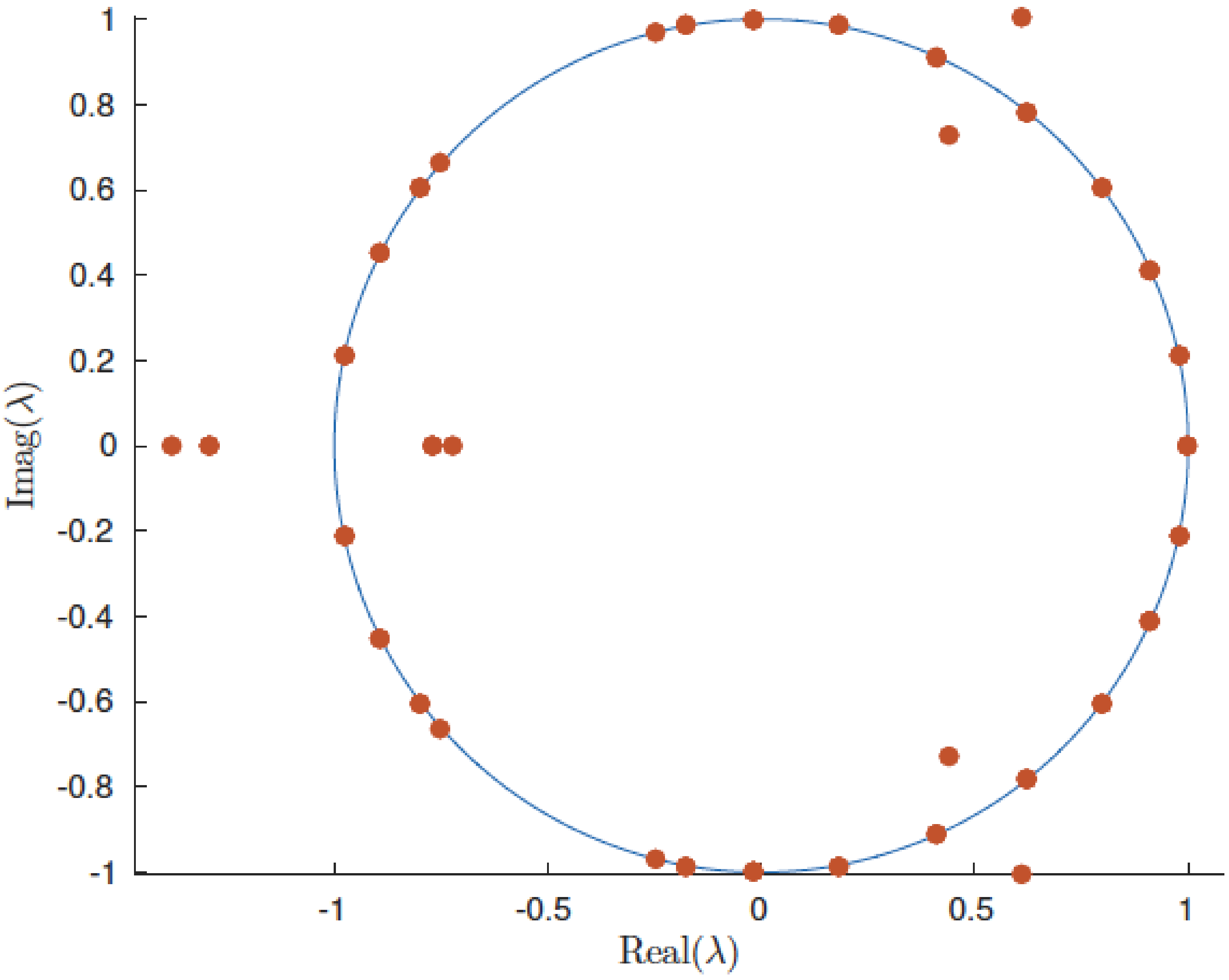


Again, note the P-T symmetry of the microstructure.

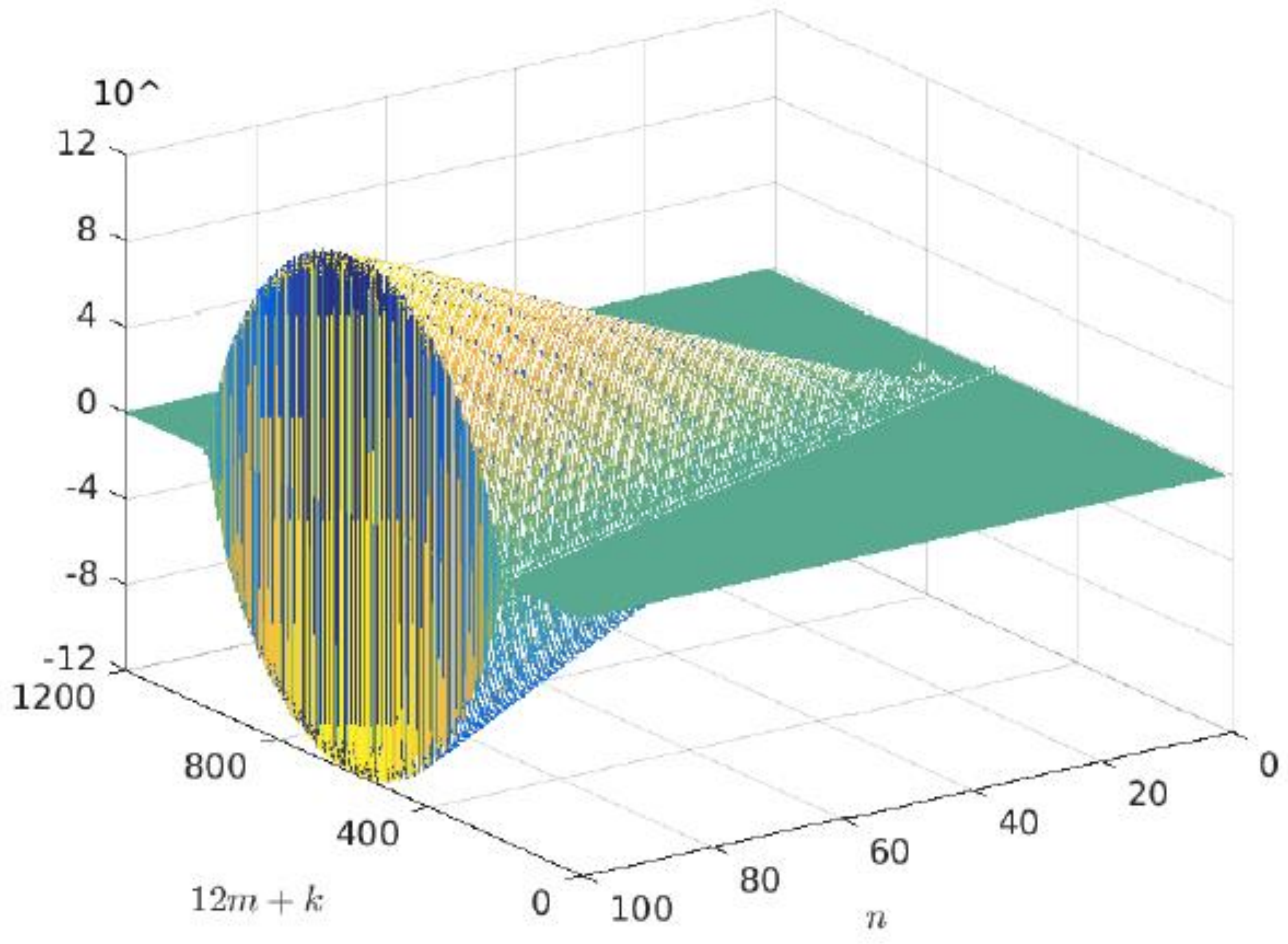
Geometry: Relation to Characteristic Lines



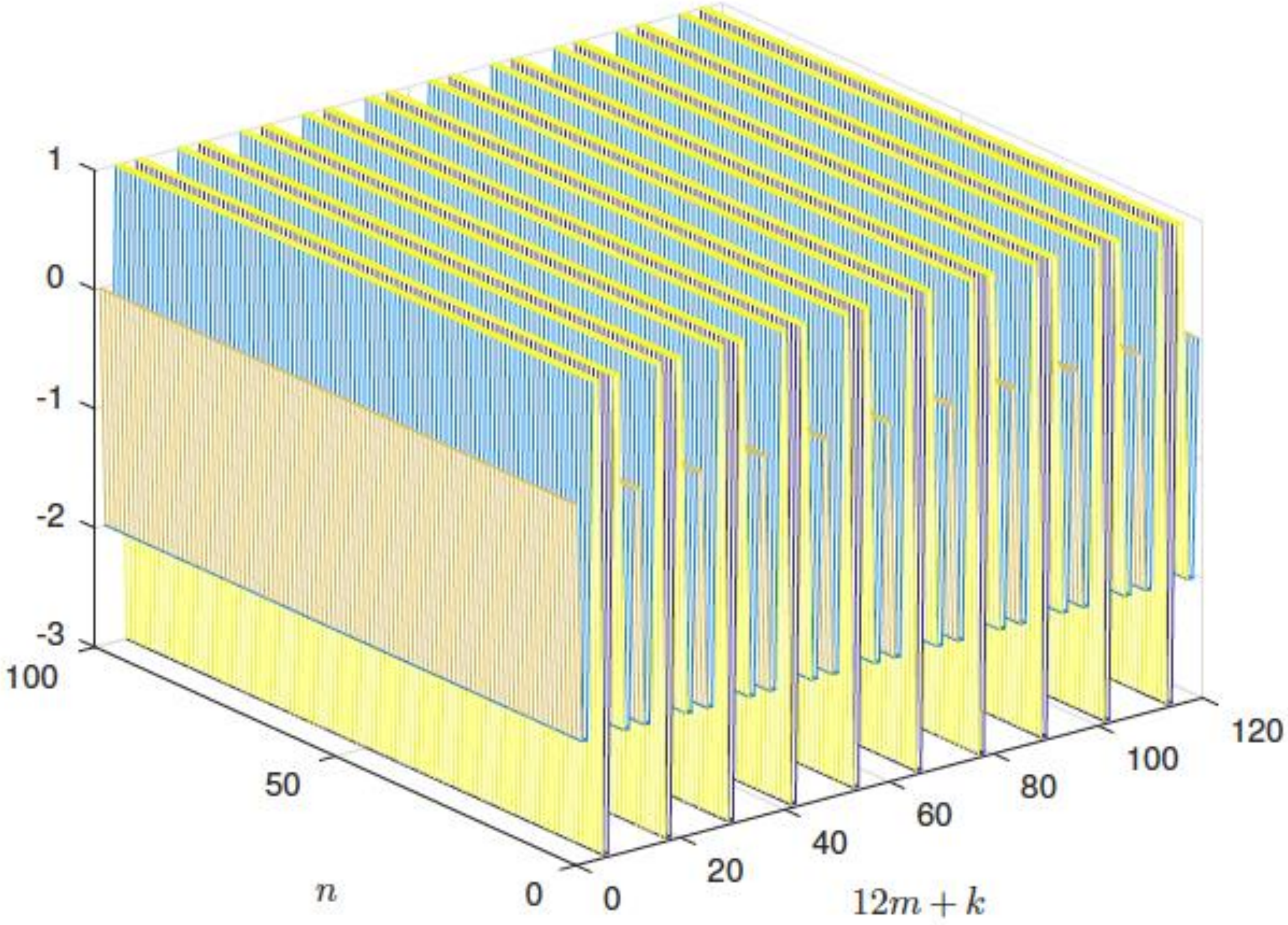
Eigenvalues of the transfer matrix



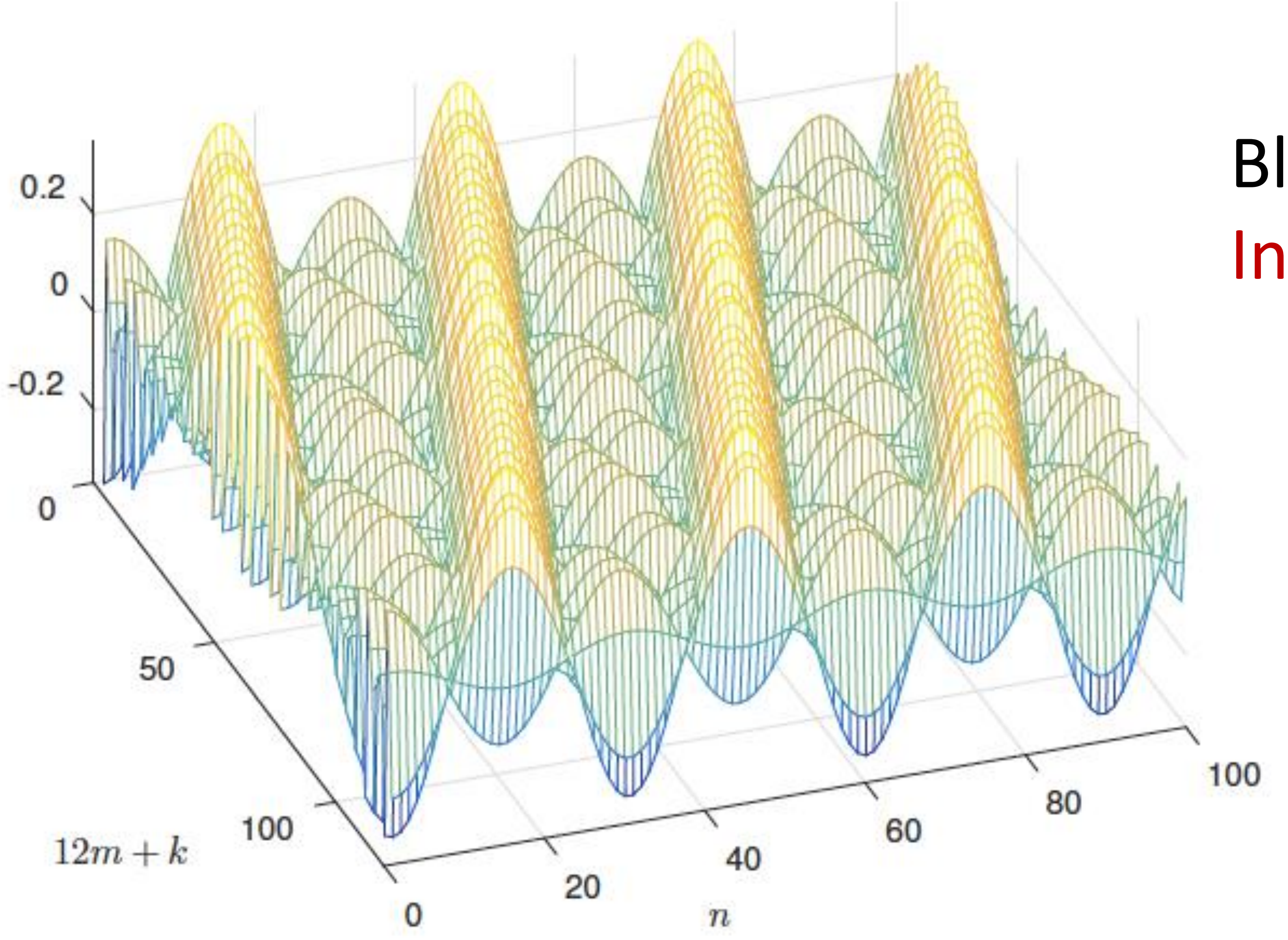
Blow up



Periodic solution



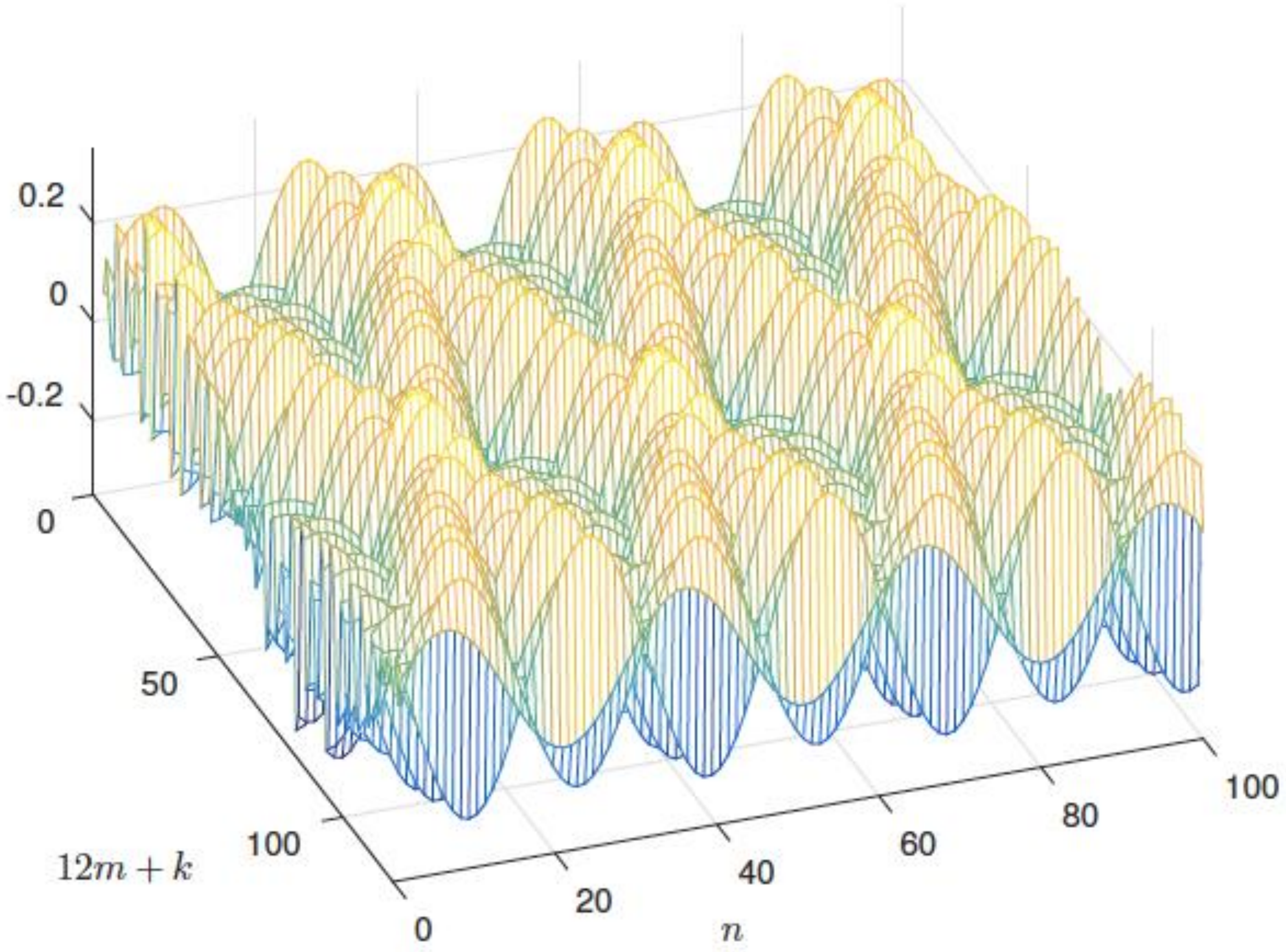
An example of a solution that does not blow up



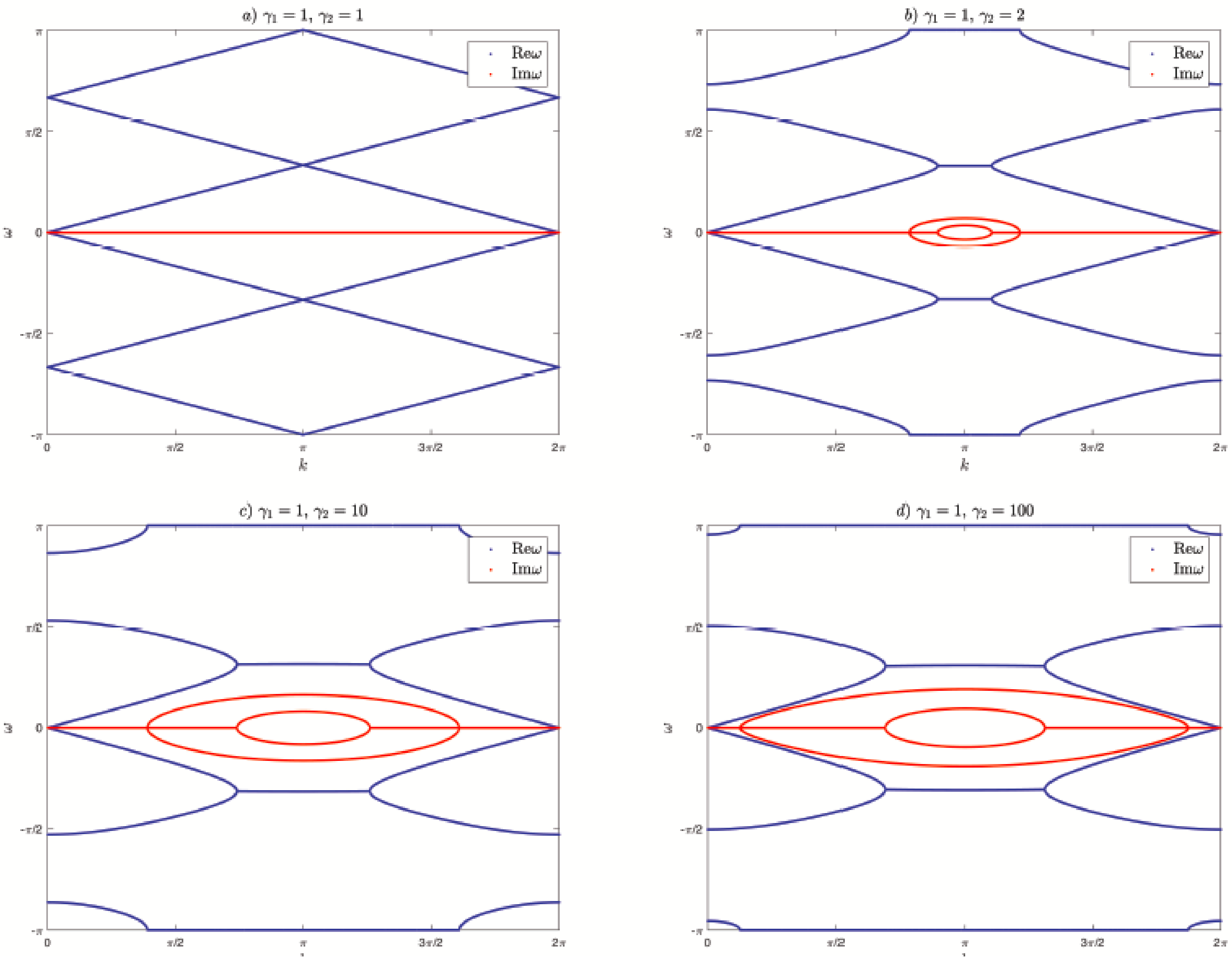
Bloch Wave:

Infinitely Degenerate!

One more solution that does not blow up

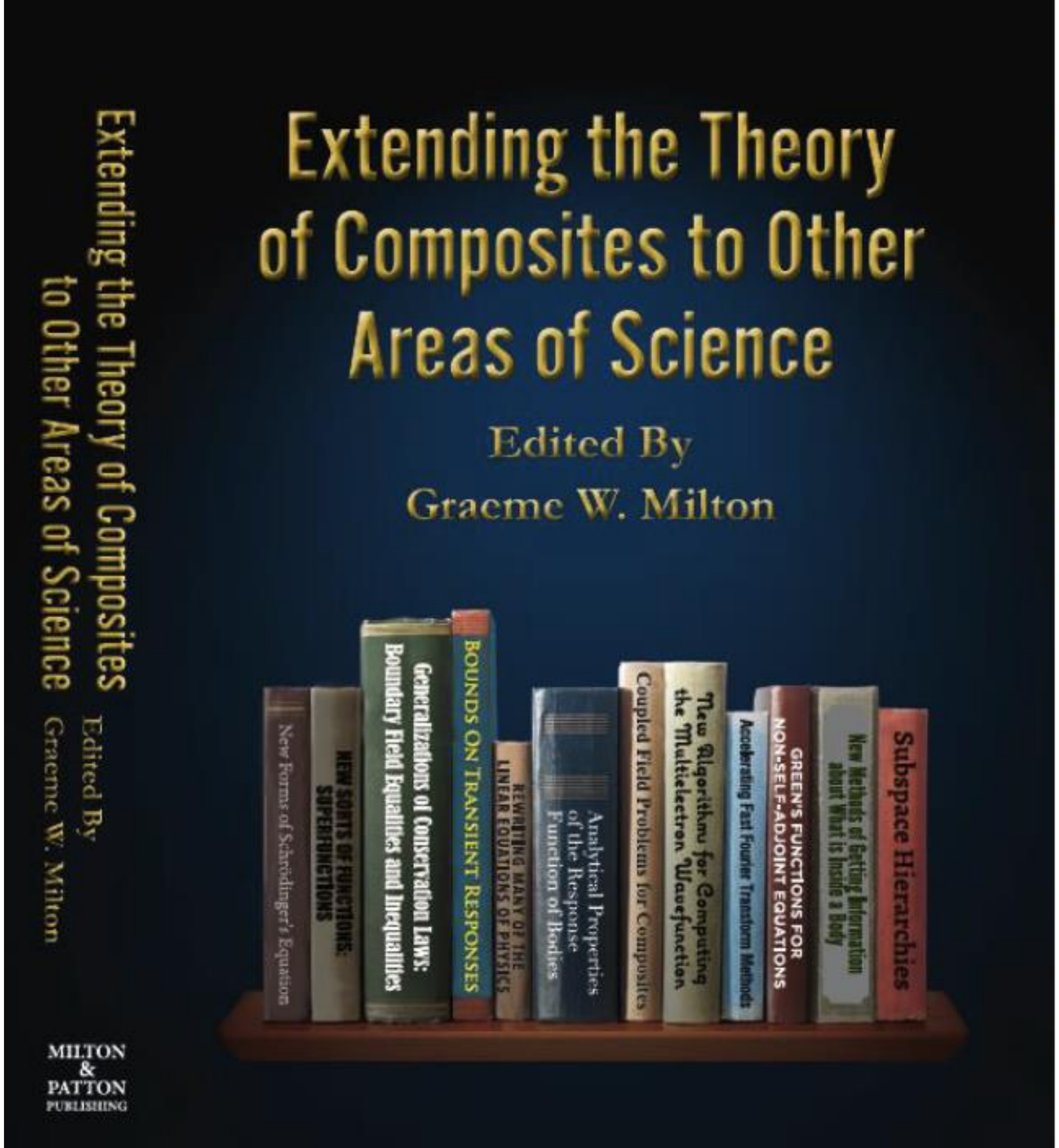


Dispersion diagrams for the microstructure with inclusions



Thank you for your attention!!

- Milton GW, Mattei O, 2017. *Field patterns: a new mathematical object*. Proc R Soc A. 473:20160819.
- Mattei O, Milton GW, 2017. *Field patterns without blow up*. To appear in New J Phys.
- Mattei O, Milton GW, 2017. *Field patterns: a new type of wave with infinitely degenerate band structure*. Submitted.



Chapters coauthored with:

Maxence Cassier
Ornella Mattei
Moti Milgrom
Aaron Welters

Available on my website

http://www.math.utah.edu/~milton/

Only $80.