The theory of field patterns

Graeme Milton

Department of Mathematics, University of Utah

Part 1: **Main Idea**, Joint work with Ornella Mattei

Part 2: **Field Patterns in Temporal Laminates**,
Joint work with Alexander Movchan, Natasha Movchan, and
Hoai-Minh Nguyen

Outline

This talk is about a new mathematical object- a new sort of wave

Space-time microstructures

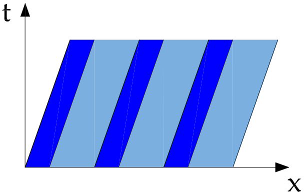
$$(a\,u_t)_t - (b\,u_x)_x = 0$$

Static materials: $a = a(x)$ and $b = b(x)$

Space-time microstructures: $a = a(x, t)$ and $b = b(x, t)$

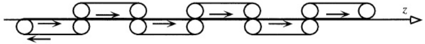
Activated materials:

The property pattern moves



Kinetic materials:

The material itself moves



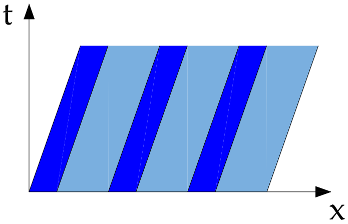
[K.A. Lurie, An Introduction to the Mathematical Theory of Dynamic Materials (2007)]

Realization of space-time microstructures

- Liquid crystals
- Ferroelectric, ferromagnetic materials
- Pump wave + small amplitudes waves: parametric resonance [e.g. Louisell & Quate (1958)]
- Transmission line with modulated inductance [e.g. Cullen (1958)]
- Experiments and more references in [Honey & Jones (1958)]

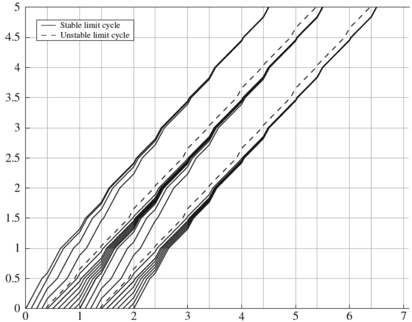
  ...
- Walking droplets [e.g. Couder et al. (2005), Couder & Fort (2006), Bush (2015)]
- Breaking reciprocity, artifical magnetism for photons [e.g. Fang et al. (2012), Boada et al. (2012), Celi et al. (2014), Yuan et al. (2016)]
- Time reversal [e.g. Fink (2016), Goussev et al. (2016)]

An example: space-time laminates



- Screening from long wave disturbances [Lurie (1997)]
- Energy conservation for low frequency waves [Lurie & Weekes (2003)]
- Energy exponential growth for high frequency waves [Cassedy (1967)]
- Homogenization for low frequencies [Lurie (1997)]

Another example: space-time checkerboards



[Lurie (2007)]

- Limit cycles + energy exponential growth [Lurie & Weekes (2006), Lurie et al. (2009)]
- Linear shocks → Quantum mechanics???
- No homogenization in the classic sense!

What are field patterns?

Field patterns arise in wave equations with a space-time microstructure, when the microstructure has the interesting feature that a disturbance propagating along a characteristic line, and subsequently interacting with the microstructure, does not evolve into a cascade of disturbances, but rather concentrates on a pattern of characteristic lines. This pattern is the field pattern!

Statement of an equivalent "conductivity" problem

2D Conductivity problem

$$\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \text{where} \quad \nabla \cdot \mathbf{j} = 0, \quad \mathbf{e} = -\nabla V,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \chi(\mathbf{x})\boldsymbol{\sigma}_1 + [1 - \chi(\mathbf{x})]\boldsymbol{\sigma}_2$$

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\beta_1 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & -\beta_2 \end{pmatrix},$$

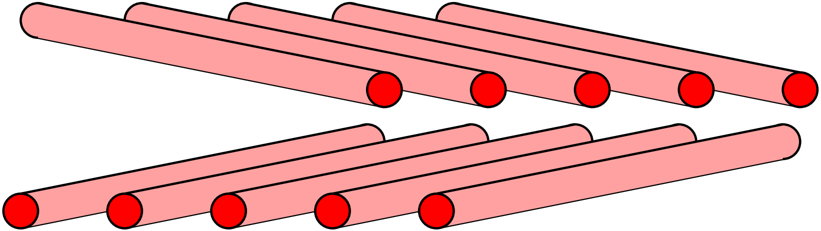
N.B. For the analogous dielectric problem–Hyperbolic materials!! [e.g. Fisher & Gould (1969), Naik et al. (2013), Korzeb et al. (2015)]

$$\alpha_i \frac{\partial^2 V_i}{\partial x_1^2} = \beta_i \frac{\partial^2 V_i}{\partial x_2^2}$$

$$x_1 \to x\,, \qquad x_2 \to t$$

$$V_i(x,t) = V_i^+(x - c_i t) + V_i^-(x + c_i t) \qquad c_i = \sqrt{\frac{\alpha_i}{\beta_i}}$$

Another way of thinking about the d'Alembert solution



Conducting wires

Transmission and initial conditions

- Transmission conditions at a space-time interface with slope w

  N.B. To have uniqueness and existence of the solution: [Lurie (1997)]

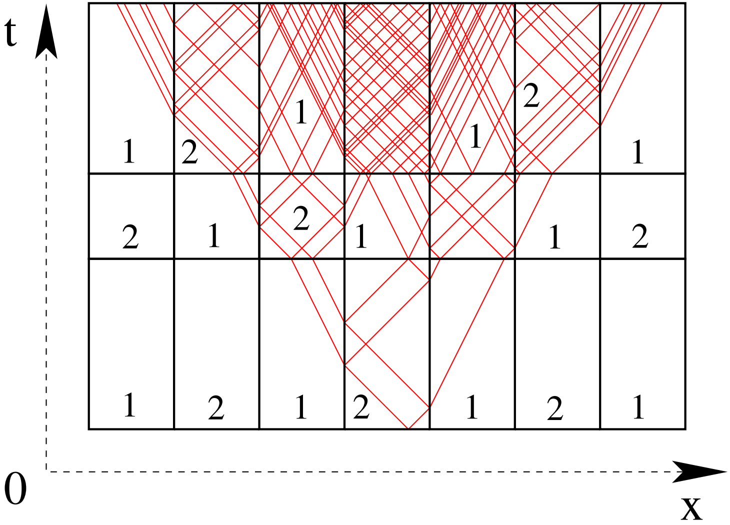
$$(w^2 - c_1^2)(w^2 - c_2^2) \geqslant 0$$

$$T.C. \begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla V_1 = \mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla V_2 \end{cases}$$

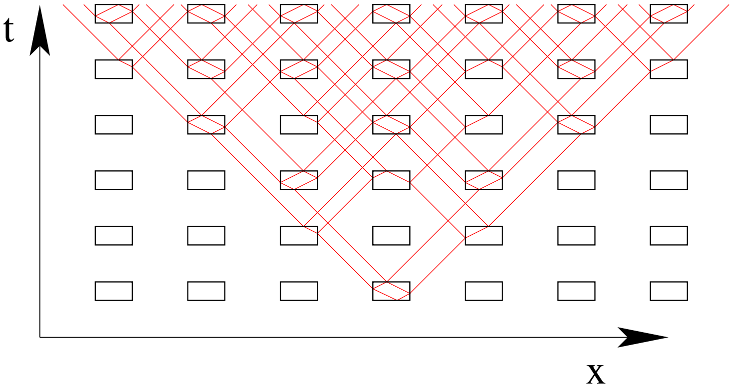
- Initial conditions

$$I.C. \begin{cases} V(x,0) = H(x-a) \\ j_2(x,0) = \delta(x-a) j_0 \end{cases}$$

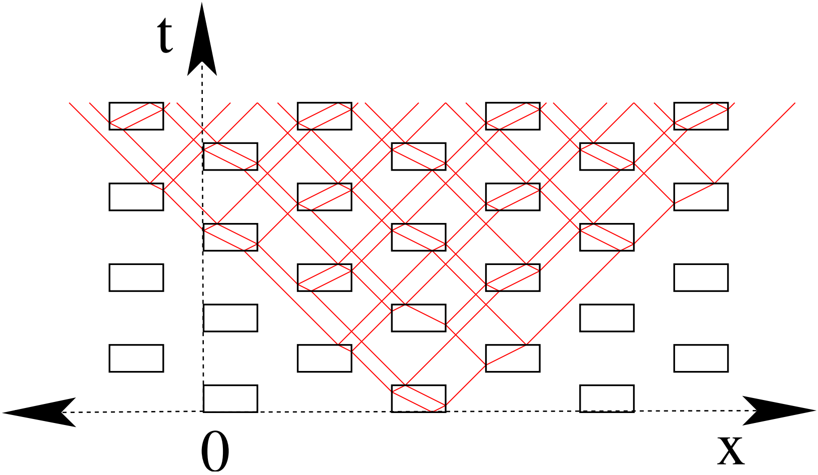
Green function for a generic space-time microstructure



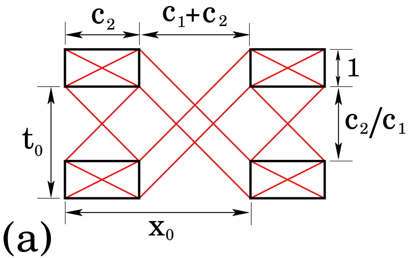
Green function for a special microstructure



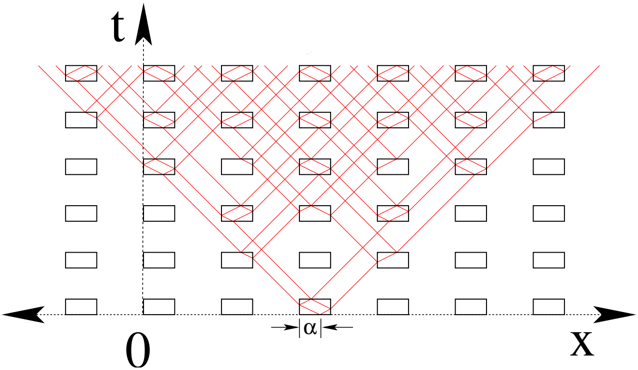
Green function for another special microstructure



Geometry: Relation to Characteristic Lines



Multidimensional nature of field patterns

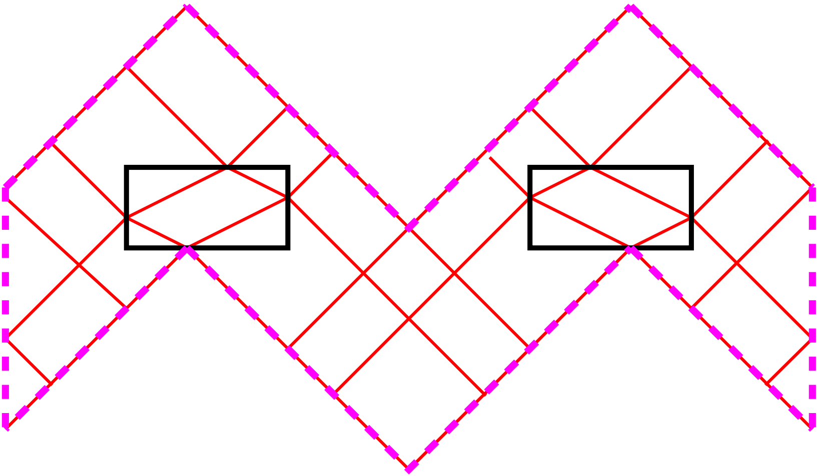


$$V(x,t) = \sum_{i=1}^{m} V_{\alpha_i}(x,t)$$

Multidimensional space: $V(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} V_{\alpha_i}(x_i, t)$

Multidimensional potential: $\mathbf{V}(x,t)$

The unit cell of the microstructure with aligned inclusions



The unit cell problem

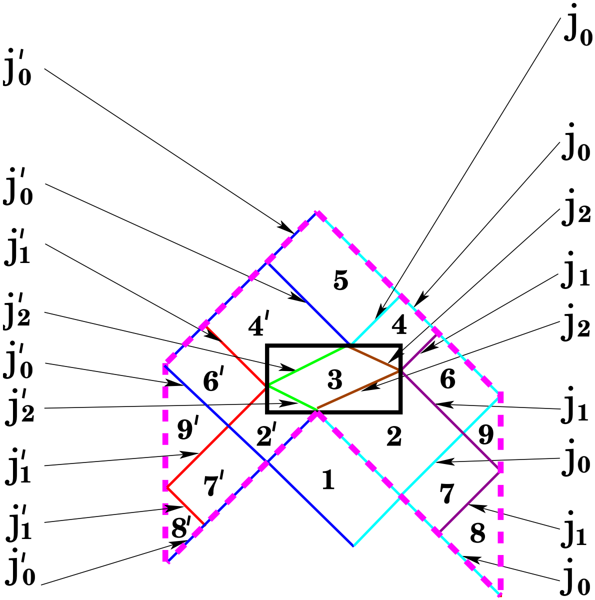
$$V_i^+(x,t) = a_i^+[1 - H(x - c_i t)], \quad V_i^-(x,t) = a_i^- H(x + c_i t)$$

$$\mathbf{j}_i^+ = a_i^+ \sqrt{\alpha_i \beta_i} \begin{pmatrix} c_i \\ 1 \end{pmatrix} \delta(x - c_i t) \equiv a_i^+ \gamma_i \frac{1}{\sqrt{1 + c_i^2}} \begin{pmatrix} c_i \\ 1 \end{pmatrix} \delta(x - c_i t)$$

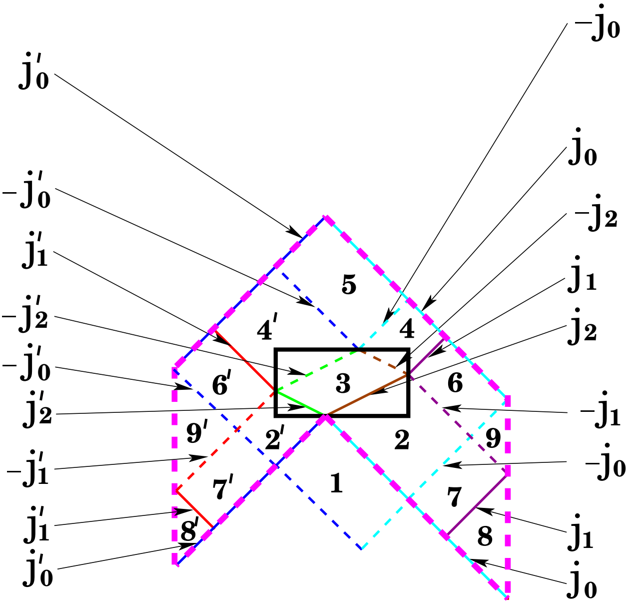
$$\mathbf{j}_i^- = a_i^- \sqrt{\alpha_i \beta_i} \begin{pmatrix} -c_i \\ 1 \end{pmatrix} \delta(x + c_i t) \equiv a_i^- \gamma_i \frac{1}{\sqrt{1 + c_i^2}} \begin{pmatrix} -c_i \\ 1 \end{pmatrix} \delta(x - c_i t)$$

with $\gamma_i = \sqrt{\alpha_i(\alpha_i + \beta_i)}$

Symmetric dynamics



Antisymmetric dynamics



"Effective properties"

"Effective conductivity tensor":

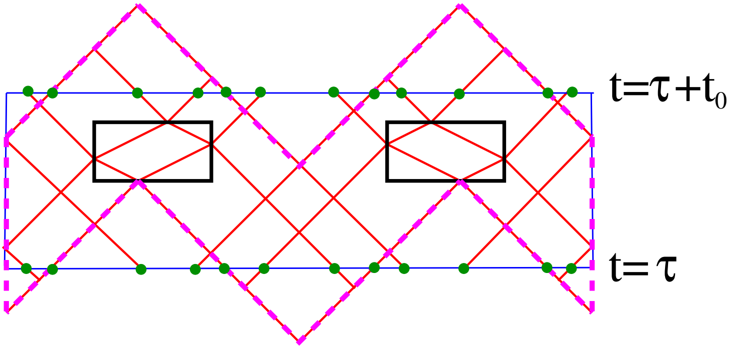
$$\boldsymbol{\sigma}_* = \begin{pmatrix} \alpha_* & 0 \\ 0 & -\beta_* \end{pmatrix} = \begin{pmatrix} \frac{c_1(c_1+2c_2)(\gamma_1+\gamma_2)}{\gamma_1^2(c_1+c_2)} & 0 \\ 0 & -\frac{(c_1+c_2)[2+(\gamma_2/\gamma_1)]}{c_1(c_1+2c_2)(\gamma_1+\gamma_2)} \end{pmatrix}$$

"Effective speed":

$$c_* = \sqrt{\alpha_*/\beta_*} = \frac{c_1(c_1+2c_2)(\gamma_1+\gamma_2)}{c_1+c_2}\sqrt{\frac{1}{\gamma_1(2\gamma_1+\gamma_2)}}$$

Homogenized equation: $\nabla \cdot \boldsymbol{\sigma}_* \nabla \underline{V} = 0$???

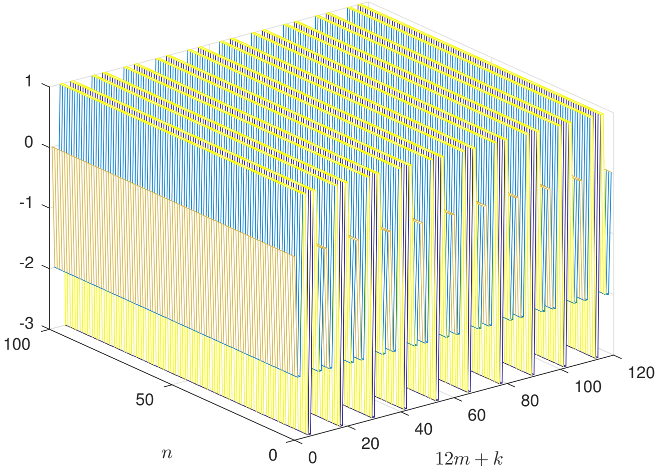
Numerical results: Transfer Matrix



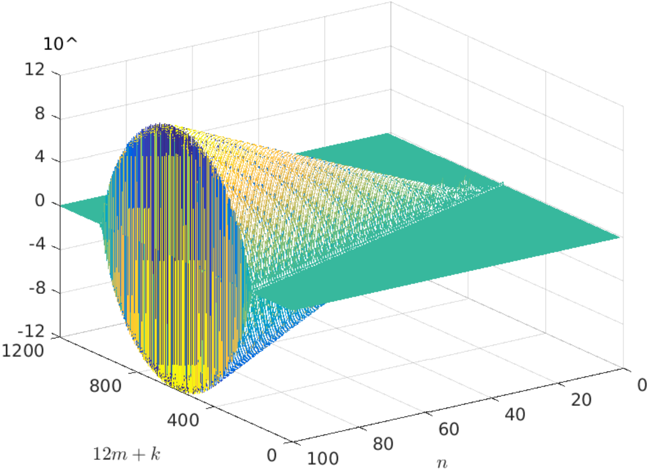
$$j(k,m,n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k',m',n)$$

$$T_{(k,m),(k,'m')} = G_{k,k'}(m-m')$$

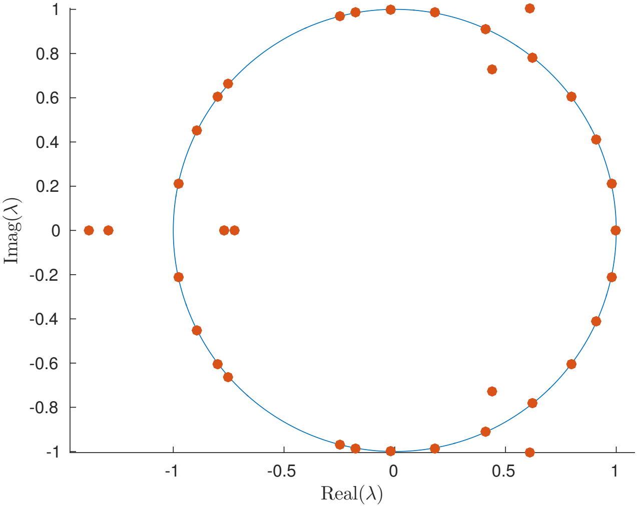
Periodic solution



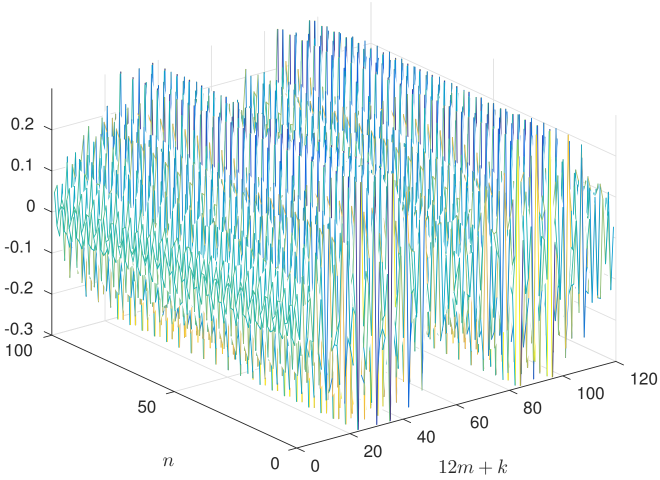
Blow up



Eigenvalues of the transfer matrix

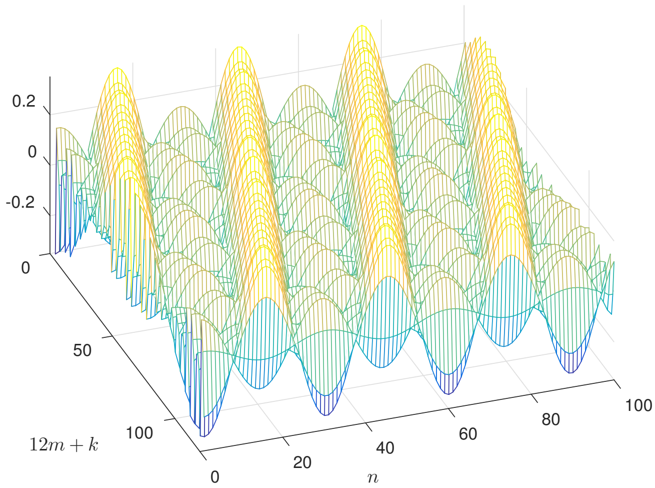


An example of a solution that does not blow up

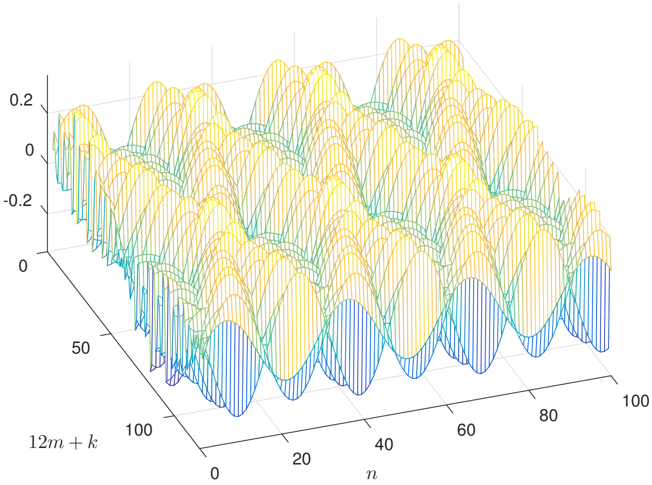


Periodic, but periodicity greater than that of the field pattern.

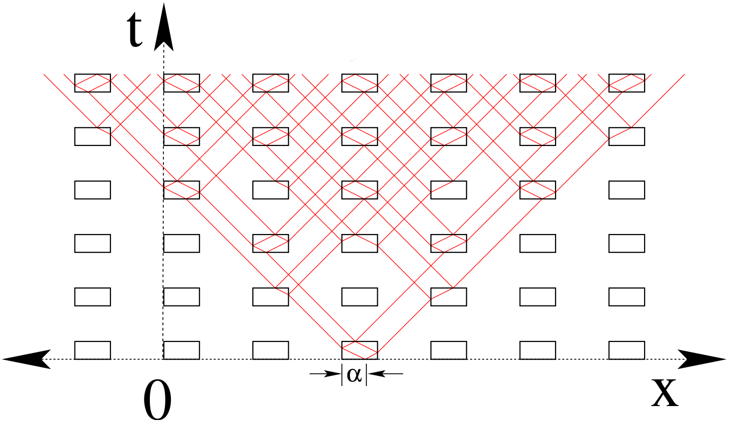
Another solution that does not blow up



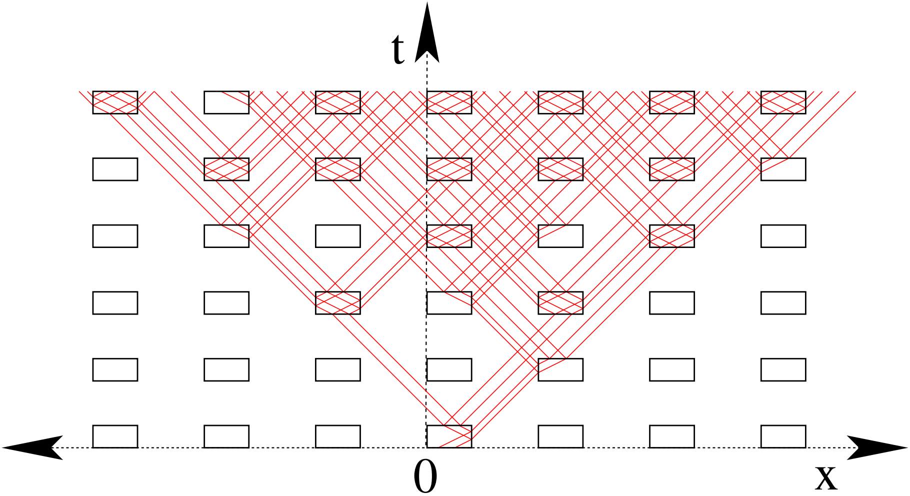
One more solution that does not blow up



Associated field patterns



Associated field patterns



Associated field patterns

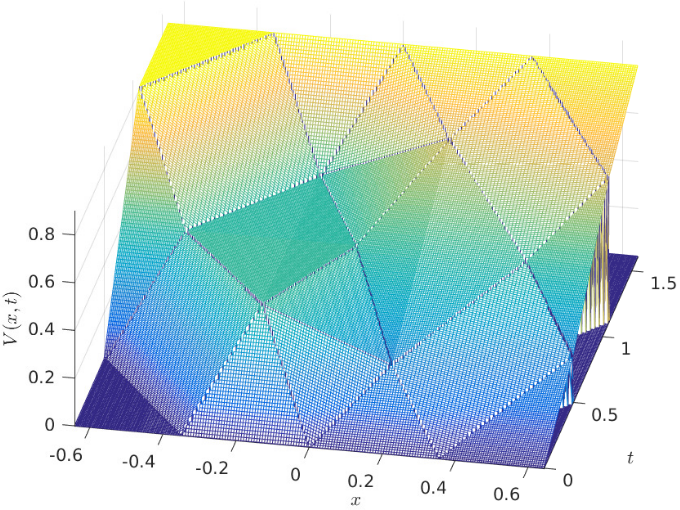
- Associated field patterns of the first degree:

$$W(x,t,\alpha_1,\alpha_2) = \int_{\alpha_1}^{\alpha_2} V(x,t,\alpha)\ d\alpha$$

- Associated field patterns of the second degree:

$$Y(x,t,\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22}) = \int_{\alpha_{11}}^{\alpha_{12}} d\alpha_1 \int_{\alpha_{21}}^{\alpha_{22}} d\alpha_2\ W(x,t,\alpha_1,\alpha_2).$$

Associated field patterns



Part 2: Field patterns in temporal laminates

Main ideas due to Alexander and Natasha Movchan and Hoai Minh Nguyen

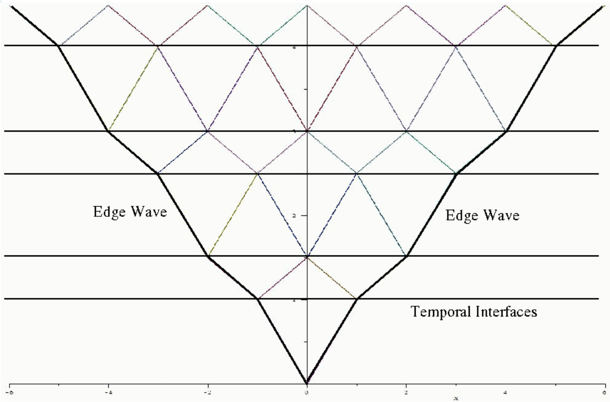


Figure: Wave split at temporal interfaces

Exponential growth of the edge wave

For $2n+1$ layers, including $n+1$ of Ω_1-type and n of Ω_2-type, the "edge wave" coefficient is equal to

$$\mathcal{C}_n = \frac{1}{2}\left(1 + \frac{1}{4}(\sqrt{\frac{\alpha_1\beta_1}{\alpha_2\beta_2}} + \sqrt{\frac{\alpha_2\beta_2}{\alpha_1\beta_1}} - 2)\right)^n, \qquad (1)$$

which grows exponentially, as $n \to \infty$ for all cases where the positive coefficients α and β are chosen in such a way that $\alpha_1\beta_1 \neq \alpha_2\beta_2$. The graphs of $\mathcal{C}_n$ for different values of the contrast parameter $\kappa = \frac{\alpha_1\beta_1}{\alpha_2\beta_2}$ are shown in the Figure below.

Edge wave amplitude

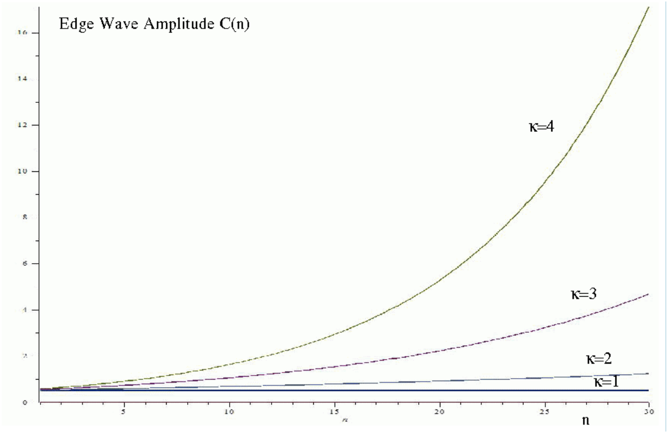


Figure: Edge wave amplitude for different values of the contrast parameter κ.

Future work

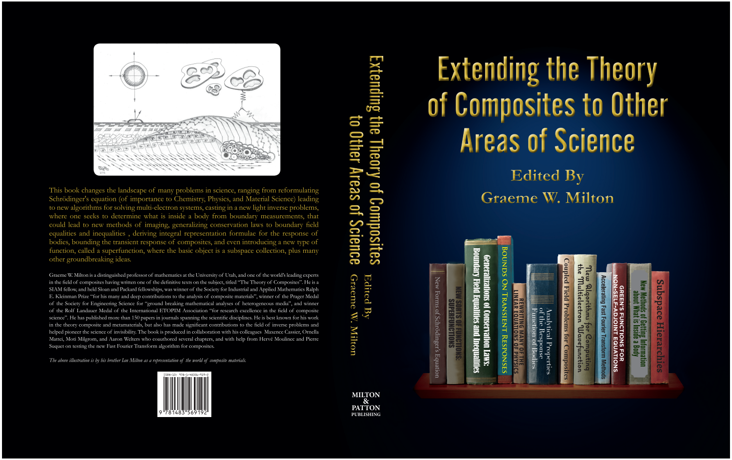
- Add a small non-linearity
- Add a small imaginary part to $\boldsymbol{\sigma}(\mathbf{x})$
- 2D + time, 3D + time
- Other wave equations
- Effective equation

The million dollar question

Are the fundamental objects in the universe, not particles, not waves, but field patterns?

Thank you for your attention!!

New book



14 chapters; 4 coauthored with Maxence Cassier, Ornella Mattei, Moti Milgrom, and Aaron Welters

Only $ 80.00, Available at `http://www.math.utah.edu/~milton/`

Green function for the aligned geometry (1)

$$j(1,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(9,1,-1)=1;\ G(10,1,-1)=\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(12,1,-1)=-\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} \end{cases}$$

$$j(2,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(1,2,0)=-\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2};\ G(3,2,0)=\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(6,2,0)=1 \end{cases}$$

$$j(3,2,0)=1 \quad \Rightarrow \quad G(11,3,-1)=1$$

$$j(4,2,0)=1 \quad \Rightarrow \quad G(8,4,0)=1$$

$$j(5,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(1,5,0)=1;\ G(4,5,0)=\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(6,5,0)=-\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} \end{cases}$$

Green function for the aligned geometry (2)

$$j(6,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(7,6,0) = -\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2};\ G(9,6,0) = \dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(10,6,0)=1 \end{cases}$$

$$j(7,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(3,7,0)=1;\ G(4,7,0) = \dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(6,7,0) = -\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} \end{cases}$$

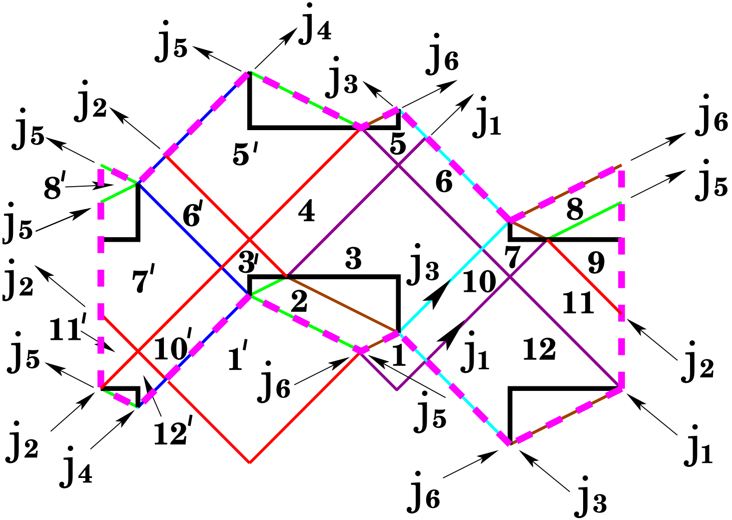
$$j(8,2,0)=1 \quad \Rightarrow \quad \begin{cases} G(7,8,0) = -\dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2};\ G(9,8,0) = \dfrac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}; \\ G(12,8,0)=1 \end{cases}$$

$$j(9,2,0)=1 \quad \Rightarrow \quad G(5,9,0)=1$$

$$j(10,2,0)=1 \quad \Rightarrow \quad G(2,10,1)=1$$

....

Symmetric dynamics for the staggered geometry



Antisymmetric dynamics for the staggered geometry