INDUCED REPRESENTATIONS FOR FINITE GROUPS

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1. Frobenius Reciprocity

1.1. **Restriction functor.** Let G be a a finite group. Let H be the a subgroup of G. Denote by $\mathcal{R}ep(G)$, resp. $\mathcal{R}ep(H)$, the categories of representations of G, resp. H.

Let (π, V) be a representation in $\mathcal{R}ep(G)$. Denote by ν the restriction of function $\pi : G \longrightarrow \mathrm{GL}(V)$ to H. Then (ν, V) is a representation in $\mathcal{R}ep(H)$. This representation is called the *restriction* of π to H and denoted by $\mathrm{Res}_H^G(\pi)$ (when there is no ambiguity we shall just write $\mathrm{Res}(\pi)$).

Clearly, Res_H^G is an exact functor form the abelian category $\mathcal{R}ep(G)$ into the abelian category $\mathcal{R}ep(H)$.

1.2. **Induction functor.** Let (ν, U) be a representation of H. Denote by $V = \mathrm{Ind}(U)$ the space of all functions $F : G \longrightarrow U$ such that $F(hg) = \nu(h)F(g)$ for all $h \in H$ and $g \in G$. Let F be the function in V and $g \in G$. Then the function $\rho(g)F : G \longrightarrow U$ defined by $(\rho(g)F)(g') = F(g'g)$ for all $g' \in G$, satisfies

$$(\rho(g)F)(hg') = F(hg'g) = \nu(h)F(g'g) = \nu(h)(\rho(g)F)(g')$$

for all $h \in H$ and $g' \in G$. Therefore $\rho(g)F$ is a function in V.

Clearly $\rho(g)$ is a linear operator on V for any $g \in G$. Moreover, $\rho(1)$ is the identity on V. For any F in V we have

$$(\rho(gg')F)(g'') = F(g''gg') = (\rho(g')F)(g''g) = (\rho(g)(\rho(g')F))(g'')$$

for all $g'' \in G$, i.e., we have

$$\rho(gg')F = \rho(g)(\rho(g')F)$$

for $g, g' \in G$. Therefore, $\rho(gg') = \rho(g)\rho(g')$ for any $g, g' \in G$ and ρ is a representation of G on V.

The representation (ρ, V) of G is called the *induced representation* and denoted by $\mathrm{Ind}_H^G(\nu)$.

If H is the identity subgroup and ν is the trivial representation, the corresponding induced representation is the regular representation of G.

Let (ν, U) and (ν', U') be two representations of H and ϕ a morphism of ν into ν'. Let F be a function in $\mathrm{Ind}(U)$. Then $\Phi(F)(g) = \phi(F(g))$ for all $g \in G$ is a function from G into U'. Moreover, we have

$$\Phi(F)(hg) = \phi(F(hg)) = \phi(\nu(h)F(g)) = \nu'(h)\phi(F(g)) = \nu'(h)\Phi(F)(g)$$

for all $h \in H$ and $g \in G$. Hence, $\Phi(F)$ is in $\mathrm{Ind}(U')$. Clearly, Φ is a linear map from $\mathrm{Ind}(U)$ into $\mathrm{Ind}(U')$.

Moreover, we have

$$(\rho'(g)\Phi(F))(g') = \Phi(F)(g'g) = \phi(F(g'g)) = \phi((\rho(g)F)(g')) = \Phi(\rho(g)F)(g')$$

for all $g' \in G$. Therefore, $\rho'(g) \circ \Phi = \Phi \circ \rho(g)$ for all $g \in G$, and Φ is a morphism of $\mathrm{Ind}_H^G(\nu)$ into $\mathrm{Ind}_H^G(\nu')$. We put $\mathrm{Ind}_H^G(\phi) = \Phi$. It is straightforward to check that in this way Ind_H^G becomes an additive functor from $\mathcal{R}ep(H)$ into $\mathcal{R}ep(G)$.

We call $\mathrm{Ind}_H^G : \mathcal{R}ep(H) \longrightarrow \mathcal{R}ep(G)$ the *induction functor.*

The next result is a functorial form of *Frobenius reciprocity.*

1.1. **Theorem.** *The induction functor* $\mathrm{Ind}_H^G : \mathcal{R}ep(H) \longrightarrow \mathcal{R}ep(G)$ *is a right adjoint functor of the restriction functor* $\mathrm{Res}_H^G : \mathcal{R}ep(G) \longrightarrow \mathcal{R}ep(H)$.

Proof. Let (ν, U) a representation of H. Consider the induced representation $\mathrm{Ind}_H^G(\nu)$ of G. The evaluation map $e : \mathrm{Ind}(U) \longrightarrow U$ given by $e(F) = F(1)$ for $F \in \mathrm{Ind}(U)$, satisfies

$$e(\rho(h)F)(1) = (\rho(h)F)(1) = F(h) = \nu(h)F(1) = \nu(h)e(F)$$

for all $F \in \mathrm{Ind}(U)$, i.e., e is a morphism of representations of H.

Let (π, V) be a representation of G. Let $\Psi : V \longrightarrow \mathrm{Ind}(U)$ be a morphism of representations of G. Then the composition $e \circ \Psi$ is a morphism of $\mathrm{Res}_H^G(\pi)$ into ν. Denote the linear map $\Psi \longmapsto e \circ \Psi$ from $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ into $\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$ by A.

Let $\phi : V \longrightarrow U$ be a morphism of representations of H. Let $v \in V$. Then we consider the function $F_v : G \longrightarrow U$ given by $F_v(g) = \phi(\pi(g)v)$ for any $g \in G$. First, for $h \in H$, we have

$$F_v(hg) = \phi(\pi(hg)v) = \phi(\pi(h)\pi(g)v) = \nu(h)\phi(\pi(g)v) = \nu(h)F_v(g)$$

for all $g \in G$. Hence F_v is a function in $\mathrm{Ind}(U)$. Consider the map $\Phi : V \longrightarrow \mathrm{Ind}(U)$ defined by $\Phi(v) = F_v$. Clearly,

$$\begin{aligned}\Phi(v + v')(g) = F_{v+v'}(g) = \phi(\pi(g)(v + v')) = \phi(\pi(g)v) + \phi(\pi(g)v')\\ = F_v(g) + F_{v'}(g) = \Phi(v)(g) + \Phi(v')(g)\end{aligned}$$

for any $g \in G$, hence we have $\Phi(v+v') = \Phi(v)+\Phi(v')$ for all $v, v' \in V$. In addition,

$$\Phi(\alpha v)(g) = \alpha\phi(\pi(g)v) = \alpha\Phi(v)(g)$$

for all $g \in G$, hence we have $\Phi(\alpha v) = \alpha\Phi(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. It follows that Φ is a linear map from V into $\mathrm{Ind}(U)$. Moreover, we have

$$\Phi(\pi(g)v)(g') = \phi(\pi(g')\pi(g)v) = \phi(\pi(g'g)v) = \Phi(v)(g'g) = (\rho(g)\Phi(v))(g')$$

for all $g' \in V$. Hence, we have $\Phi(\pi(g)v) = \rho(g)\Phi(v)$ for all $g \in G$ and $v \in V$. Therefore, Φ is a morphism of representations (π, V) and $\mathrm{Ind}_H^G(\nu)$ of G. Denote the map $\phi \longmapsto \Phi$ from $\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$ into $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ by B.

Clearly, for $\phi \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$, we have

$$((A \circ B)(\phi))(v) = (A(\Phi))(v) = \Phi(v)(1) = F_v(1) = \phi(v)$$

for all $v \in V$. Therefore, $A \circ B$ is the identity map.

In addition, for $\Psi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$, we have

$$\begin{aligned}(((B \circ A)(\Psi))(v))(g) = (B(A(\Psi))(v))(g) = A(\Psi)(\pi(g)v)\\ = (\Psi(\pi(g)v))(1) = (\rho(g)\Psi(v))(1) = \Psi(v)(g)\end{aligned}$$

for all $g \in G$. Hence, we have $((B\circ A)(\Psi))(v) = \Psi(v)$ for all $v \in V$, i.e., $(B\circ A)(\Psi) = \Psi$ for all Ψ and $B \circ A$ is also the identity map. □

By Maschke's theorem, $\mathcal{R}ep(H)$ is semisimple, and every short exact sequence splits. Therefire we have the following result.

1.2. **Theorem.** *The induction functor* $\mathrm{Ind}_H^G : \mathcal{R}ep(H) \longrightarrow \mathcal{R}ep(G)$ *is exact.*

1.3. **Induction in stages.** Let K be a subgroup of H. Then we have $\mathrm{Res}_K^G = \mathrm{Res}_K^H \circ \mathrm{Res}_H^G$ as functors from $\mathcal{R}ep(G)$ into $\mathcal{R}ep(K)$. Since induction functors are right adjoints, this immediately implies the following result which is called the *induction in stages.*

1.3. **Theorem.** *Let H be a subgroup of G and K a subgroup of H. Then the functors* Ind_K^G *and* $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H$ *are isomorphic.*

1.4. **Frobenius Reciprocity.** Obviously, the restriction functor Res_H^G maps finite-dimensional representations into finite dimensional representations. From the following result we see that the induction functor Ind_H^G does the same.

1.4. **Proposition.** *Let (ν, U) be a finite-dimensional representation of H. Then*

$$\dim \mathrm{Ind}_H^G(\nu) = \mathrm{Card}(H\backslash G) \cdot \dim(\nu).$$

Proof. Let C be a right H-coset in G. Let g_C be an element in C. Then the functions

$$F_{C,v}(g) = \begin{cases} \nu(gg_C^{-1})v & \text{for } g \in Hg_C; \\ 0 & \text{for } g \notin Hg_C; \end{cases}$$

span $\mathrm{Ind}(U)$. If $e_1, e_2, \ldots, e_m$ is a basis of U, the family F_{C,e_i}, $C \in H\backslash G$, $1 \leq i \leq m$, is a basis of $\mathrm{Ind}(U)$. □

Let (π, V) be an irreducible representation of G and ν an irreducible representation of H. Then $\mathrm{Ind}_H^G(\nu)$ is finite-dimensional by 1.4 and a direct sum of irreducible representations of G. The multiplicity of π in this direct sum is $\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ by Schur Lemma. By 1.1, we conclude that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu)) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu).$$

The latter expression is the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$.

This leads to the following version of Frobenius reciprocity for representations of finite groups.

1.5. **Theorem.** *Let π be an irreducible representation of G and ν an irreducible representation of H. Then the multiplicity of π in $\mathrm{Ind}_H^G(\nu)$ is equal to the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$.*

1.5. **An example.** Let S_3 be the symmetric group in three letters. We shall show how above results allow us to construct irreducible representations of S_3.

The order of S_3 is $3! = 6$. It contains the normal subgroup A_3 consisting of all even permutations which is of order 3. The quotient group S_3/A_3 consists of two elements.

The identity element is $(1\ 2\ 3)$. The other two even permutations are $(2\ 3\ 1)$ and $(3\ 1\ 2)$. We have $(2\ 1\ 3)^2 = 1$ and

$$(2\ 1\ 3)(2\ 3\ 1)(2\ 1\ 3) = (3\ 1\ 2).$$

Hence nontrivial even permutations form a conjugacy class.

The odd permutations are $(2\ 1\ 3)$, $(1\ 3\ 2)$ and $(3\ 2\ 1)$. Since $(2\ 1\ 3)(1\ 3\ 2)(2\ 1\ 3) = (3\ 1\ 2)$, $(1\ 3\ 2)$ and $(3\ 2\ 1)$ are conjugate. On the other hand, $(1\ 3\ 2)^2 = 1$ and

(1 3 2)(2 3 1)(1 3 2) = (3 2 1), and (2 3 1) and (3 2 1) are conjugate. Therefore all odd permutations form a conjugacy class. It follows that S_3 has three conjugacy classes. Therefore S_3 has three irreducible representations.

Clearly, two irreducible representations of S_3 are the trivial representation and the sign representation. Since $1^2 + 1^2 + 2^2 = 6$, by Burnside theorem, the third irreducible representation π is two-dimensional. By **??**, the character of regular representation is 6 at the identity element and 0 on all other elements. By Burnside theorem the character of π is one half of the difference of the characters of regular representation and the direct sum of trivial and sign representation. The latter character is 2 on even elements and 0 on odd elements. Therefore, the character of π is 2 at the identity, -1 on nontrivial even elements and 0 at odd elements. It follows that the character of π is supported on A_3.

The group A_3 is cyclic with three elements. It has two nontrivial one-dimensional representations. If we pick a generator $a = (2\ 3\ 1)$ of A_3 one character maps a into $e^{i\frac{2\pi}{3}}$ and the other maps a to $e^{-i\frac{2\pi}{3}}$. We call the first one ν. By a direct calculation we see that $(2\ 1\ 3)a(2\ 1\ 3) = a^{-1}$. The restriction of π to A_3 is a direct sum of two characters of A_3. Since we know that $\operatorname{ch}(\pi)(a) = -1$ we see that it must be

$$\nu(a) + \nu(a)^{-1} = e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1.$$

Therefore, $\operatorname{Res}^{S_3}_{A_3}(\pi) = \nu \oplus \nu^{-1}$.

By Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{S_3}(\pi, \operatorname{Ind}^{S_3}_{A_3}(\nu)) = \dim \operatorname{Hom}_{A_3}(\operatorname{Res}^{S_3}_{A_3}(\pi), \nu) = 1.$$

Hence, π is a equivalent to a subrepresentation of $\operatorname{Ind}^{S_3}_{A_3}(\nu)$. Since their dimensions are equal, we have $\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu)$. Analogously, we prove that $\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})$.

Therefore we proved that the dual of S_3 consists of the classes of the trivial representation, sign representation and the induced representation $\operatorname{Ind}^{S_3}_{A_3}(\nu) \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})$.

1.6. **Characters of induced representations.** Let (ν, U) be a finite-dimensional representation of H. Let $(e_i; 1 \leq i \leq n)$ be a basis of U. In the proof of 1.4, we constructed a basis $(F_{C,i}; C \in H\backslash G,\ 1 \leq i \leq n)$ of $\operatorname{Ind}(U)$. Let $C \in H\backslash G$ and $1 \leq i \leq n$. Let $g \in G$. Then

$$(\rho(g)F_{C,i})(g') = F_{C,i}(g'g)$$

for all $g' \in G$, i.e., $\rho(g)F_{C,i}$ is supported on the coset $D = C \cdot g^{-1}$. Therefore, it is a linear combination of $F_{D,j}$, $1 \leq j \leq n$, i.e.,

$$\rho(g)F_{C,i} = \sum_{j=1}^{n} c_j F_{D,j}.$$

Hence, $\rho(g)F_{C,i}$ is a linear combination of $F_{C,j}$, $1 \leq j \leq n$, if and only if $D = C$, i.e., g_C and $g_C g$ are in the same H-coset. This implies that $g_C g = h g_C$ for some $h \in H$, i.e., $g_C g g_C^{-1} = h \in H$. Conversely, if $g_C g g_C^{-1} \in H$ for some C, we have

$$C = Hg_C = Hg_C g = C \cdot g$$

and g_C and $g_C g$ are in the same H-coset. Moreover, we have

$$(\rho(g)F_{C,i})(g_C) = F_{C,i}(g_C g) = F_{C,i}(h g_C) = \nu(h) F_{C,i}(g_C)$$
$$= \nu(h)e_i = \sum_{j=1}^{n} \nu(h)_{ji} e_j = \sum_{j=1}^{n} \nu(h)_{ji} F_{C,j}(g_C).$$

This in turn implies that

$$\rho(g)F_{C,i} = \sum_{j=1}^{n} \nu(h)_{ji} F_{C,j}$$

if $C \cdot g^{-1} = C$. Therefore, the matrix of $\rho(g)$ has a nonzero diagonal entry in the basis $(F_{C,i}, C \in H\backslash G, 1 \le i \le n)$, only if $C = C \cdot g$ and then these entries are $\nu(h)_{jj}$, $1 \le j \le n$. This implies that

$$\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g) = \sum_{C\cdot g = C} \operatorname{ch}(\nu)(h) = \sum_{C\cdot g=C} \operatorname{ch}(\nu)(g_C g g_C^{-1})$$
$$= \sum_{g_C g g_C^{-1} \in H} \operatorname{ch}(\nu)(g_C g g_C^{-1}) = \frac{1}{[H]} \sum_{h\in H} \sum_{g_C g g_C^{-1}\in H} \operatorname{ch}(\nu)(h g_C g g_C^{-1} h^{-1})$$
$$= \frac{1}{[H]} \sum_{g' g g'^{-1} \in H} \operatorname{ch}(\nu)(g' g g'^{-1}).$$

We extend the character of ν to a function χ_ν on G which vanishes outside H. Then we get the following result.

1.6. **Theorem.** *The character of induced representation* $\operatorname{Ind}_H^G(\nu)$ *is equal to*

$$\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g) = \frac{1}{[H]} \sum_{g'\in G} \chi_\nu(g' g g'^{-1}).$$

Therefore the character of the induced representation is proportional to the average of the function χ_ν on the equivalence classes in G.

In particular we have the following result.

1.7. **Corollary.** *The character of* $\operatorname{Ind}_H^G(\nu)$ *is supported in the union of conjugacy classes in* G *which intersect* H.

The result is particularly simple if H is a normal subgroup of G.

1.8. **Corollary.** *Let* H *be a normal subgroup of* G. *Then:*

(i) *the support of the character of* $\operatorname{Ind}_H^G(\nu)$ *is in* H;

(ii) *we have*

$$\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(h) = \frac{1}{[H]} \sum_{g\in G} \operatorname{ch}(\nu)(g h g^{-1})$$

for any $h \in H$.

1.7. **An example.** Consider again the representation $\pi \cong \operatorname{Ind}_{A_3}^{S_3}(\nu)$. By the above formula, its character vanishes outside of A_3 and is equal to

$$\operatorname{ch}(\pi)(h) = \frac{1}{3} \sum_{g\in S_3} \nu(g h g^{-1})$$

for $h \in A_3$. If $h = 1$, we see that

$$\operatorname{ch}(\pi)(1) = \frac{6}{3} = 2.$$

If $h = a$, we have $gag^{-1} = a$ for $g \in A_3$. If g is not in A_3, it is in the other A_3-coset. Therefore, it is in the coset represented by (2 1 3). By the calculation done before, $gag^{-1} = a^{-1}$ for $g \notin A_3$. Therefore, we have

$$\operatorname{ch}(\pi)(a) = \frac{1}{3} \sum_{g \in S_3} \nu(gag^{-1}) = \nu(a) + \nu(a^{-1}) = -1.$$

This agrees with the calculation of the character of π done before.

1.8. **Characters and Frobenius reciprocity.** Now we are going to give a proof of 1.5 based on character formula for the induced representation and the orthogonality relations.

We denote by $(\cdot \mid \cdot)_G$ the inner product on $\mathbb{C}[G]$ and by $(\cdot \mid \cdot)_H$ the inner product on $\mathbb{C}[H]$. Let π be a finite-dimensional representation of G and ν a finite-dimensional representation of H. Then we have

$$(\operatorname{ch}(\operatorname{Ind}_H^G(\nu)) \mid \operatorname{ch}(\pi))_G = \frac{1}{[G]} \sum_{g \in G} \operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g)\overline{\operatorname{ch}(\pi)(g)}$$

$$= \frac{1}{[G][H]} \sum_{g \in G} \left(\sum_{g' \in G} \chi_\nu(g' g g'^{-1}) \overline{\operatorname{ch}(\pi)(g)} \right) = \frac{1}{[H]} \sum_{g' \in G} \frac{1}{[G]} \left(\sum_{g \in G} \chi_\nu(g' g g'^{-1}) \overline{\operatorname{ch}(\pi)(g)} \right)$$

$$= \frac{1}{[H]} \sum_{g' \in G} \frac{1}{[G]} \left(\sum_{g \in G} \chi_\nu(g) \overline{\operatorname{ch}(\pi)(g)} \right) = \frac{1}{[H]} \sum_{h \in H} \operatorname{ch}(\nu)(h) \overline{\operatorname{ch}(\pi)(h)}$$

$$= (\operatorname{ch}(\nu) \mid \operatorname{ch}(\operatorname{Res}_H^G(\pi)))_H.$$