1. Representations of finite groups

1.1. Category of group representations. Let $G$ be a group. Let $V$ be a vector space over $\mathbb{C}$. Denote by $\text{GL}(V)$ the general linear group of $V$, i.e., the group of all linear automorphisms of $V$.

A representation $(\pi, V)$ of $G$ on the vector space $V$ is a group homomorphism $\pi : G \rightarrow \text{GL}(V)$. A morphism $\phi : (\pi, V) \rightarrow (\nu, U)$ of representation $(\pi, V)$ into $(\mu, U)$ is a linear map $\phi : V \rightarrow U$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\pi(g)} & V \\
\downarrow & & \downarrow \\
U & \xrightarrow{\nu(g)} & U \\
\end{array}
$$

commutes for all $g \in G$. Morphisms of representations are also called intertwining maps. The set of all morphisms of $(\pi, V)$ into $(\nu, U)$ is denoted by $\text{Hom}_G(V, U)$.

It is easy to check that all representations of $G$ form a category $\mathcal{R}ep(G)$ of representations of $G$.

An isomorphism $\phi : (\pi, V) \rightarrow (\nu, U)$ in this category is a morphism of representations which is a linear isomorphism of the vector space $V$ with $U$. If two representations are isomorphic, we say that they are equivalent.

Let $(\pi, V)$ and $(\nu, U)$ be two representations of $G$. Let $U$ be a subspace of $V$ which is invariant for $G$, i.e., $\pi(g)(U) \subset U$ for all $g \in G$. Then the linear maps $\pi(g)$ restricted to $U$ define a morphism $\phi : (\pi, V) \rightarrow (\nu, U)$.

If $(\pi, V)$ and $(\nu, U)$ are two representations of $G$, we can define the representation $\pi \oplus \nu$ of $G$ on $V \oplus V$ such that $(\pi \oplus \nu)(g)(v, u) = (\pi(g)v, \nu(g)u)$ for all $g \in G$, $v \in V$ and $u \in U$. The representation $\pi \oplus \nu$ is called the direct sum of $\pi$ and $\nu$.

Let $(\pi, V)$ be a representation of $G$. Let $U$ be a subspace of $V$ which is invariant for $G$, i.e., $\pi(g)(U) \subset U$ for all $g \in G$. Then the linear maps $\pi(g)$ restricted to $U$ define a subrepresentation of $(\pi, V)$.

Let $\phi : (\pi, V) \rightarrow (\nu, U)$ be a morphism of representations. Then, ker $\phi \subset V$ is a $G$-invariant subspace of $V$. Hence, ker $\phi$ is a subrepresentation of $(\pi, V)$.

Let $(\pi, V)$ be a representation of $G$. Let $U$ be an invariant subspace of $V$. For each $g \in G$ we define a linear operator $\rho(g)$ on the quotient space $V/U$ by $\rho(g)(v + U) = \pi(g)v + U$ for any $g \in G$. Then $(\rho, V/U)$ is a quotient representation of $(\pi, V)$.

Clearly, the category $\mathcal{R}ep(G)$ is an abelian category.
If the vector space $V$ is equipped with an inner product $(\cdot | \cdot)$ and all linear operators $\pi(g), g \in G$, are unitary with respect to this inner product structure, we say that the representation $(\pi, V)$ is unitary.

1.2. **Representations of finite groups.** Let $G$ be a group. We say that $G$ is a finite group if $G$ is a finite set.

In this section we assume that the group $G$ is finite. We put $|G| = \text{Card}(G)$.

A representation $(\pi, V)$ of $G$ is finite-dimensional if $V$ is a finite-dimensional vector space. We put $\dim \pi = \dim \mathbb{C}V$.

1.2.1. **Lemma.** Let $(\pi, V)$ be a representation of $G$. Let $v \in V$, $v \neq 0$. Then there exists a finite-dimensional subrepresentation $(\nu, U)$ of $(\pi, V)$ such that $v \in U$.

Proof. Let $U$ be the vector subspace of $V$ generated by vectors $\pi(g)v$, $g \in G$. Then $U$ is $G$-invariant and finite-dimensional. Moreover, $v = \pi(1)v$ is in $U$. $\square$

A representation $(\pi, V)$ of $G$ is called irreducible if the only $G$-invariant subspaces in $V$ are $\{0\}$ and $V$.

1.2.2. **Theorem.** Let $(\pi, V)$ be an irreducible representation of $G$. Then $\pi$ is finite-dimensional.

Proof. Let $v \in V$, $v \neq 0$. By 1.2.1, $V$ contains a finite-dimensional $G$-invariant subspace $U$ such that $v \in U$. If $\pi$ is irreducible, we must have $U = V$ and $V$ is finite-dimensional. $\square$

1.2.3. **Corollary.** Every representation $(\pi, V)$ of $G$ contains an irreducible subrepresentation.

The main result on representations of finite groups is the following observation.

1.2.4. **Theorem** (Mascke). Let $(\pi, V)$ be a representation of $G$. Let $(\nu, U)$ be a subrepresentation of $(\pi, V)$. Then there exists a subrepresentation $(\rho, W)$ of $(\pi, V)$ such that $\pi = \nu \oplus \rho$.

Proof. Let $P$ be a projector of $V$ onto $U$. Consider the linear map

$$Q = \frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) P \pi(g)$$

on $V$. Clearly, since $U$ is $G$-invariant, $Q(V) \subset U$. Moreover, for any $u \in U$, we have

$$Qu = \frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) P \pi(g)u = \frac{1}{|G|} \sum_{g \in G} u = u.$$ 

Therefore, $U = \text{im} Q$ and $Q^2 = Q$. It follows that $Q$ is a projection onto $U$ along $\ker Q$.

In addition, we have

$$Q \pi(h) = \frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) P \pi(gh) = \frac{1}{|G|} \sum_{g \in G} \pi(hg^{-1}) P \pi(g) = \pi(h)Q$$

for all $h \in G$, i.e., $Q$ is a morphism of $(\pi, V)$ into $(\nu, U)$. Hence $W = \ker Q$ is a $G$-invariant subspace and $V = U \oplus W$. $\square$

Therefore, the category $\text{Rep}(G)$ is semisimple.
1.3. **Schur Lemma.** Let \((\nu, U)\) and \((\pi, V)\) be two irreducible representations of \(G\). Let \(A\) be a morphism of \(\nu\) into \(\pi\). Then, for \(u \in \ker A\), we have \(A(\nu(g)u) = \pi(g)Au = 0\) for any \(g \in G\). Hence, \(\nu(g)u \in \ker A\) for any \(g \in G\), and \(\ker A\) is a \(G\)-invariant subspace of \(U\).

Since \(\nu\) is irreducible, \(\ker A\) is equal to either \(\{0\}\) or \(U\). In the latter case, we see that \(A = 0\). In the first case, \(A\) is injective. Moreover, if \(v \in \text{im } A\), we have \(v = Au\) for some \(u \in U\). Therefore, \(\pi(g)v = \pi(g)Au = A(\nu(g)u)\) is in \(\text{im } A\) for any \(g \in G\). Hence, \(\text{im } A\) is a \(G\)-invariant subspace of \(V\). Since \(A\) is injective, \(\text{im } A\) is not \(\{0\}\). It follows that \(\text{im } A = V\), and \(A\) is an isomorphism.

This implies the following result.

1.3.1. **Proposition.** Let \((\nu, U)\) and \((\pi, V)\) be two irreducible representations of \(G\). Assume that \(\pi\) and \(\nu\) are not equivalent. Then \(\text{Hom}_G(U, V) = \{0\}\). In addition we have the following result.

1.3.2. **Theorem (Schur Lemma).** Let \((\pi, V)\) be an irreducible representation of \(G\). Then \(\text{Hom}_G(V, V) = CI\).

**Proof.** Let \(A\) be an endomorphism of \(\pi\). Since \(V\) is finite-dimensional, \(A\) has an eigenvalue \(\lambda \in \mathbb{C}\). Therefore, \(B = A - \lambda I\) is an endomorphism of \(\pi\) which is not injective. By the above discussion, it must be equal to 0. Hence, we have \(A = \lambda I\). \(\square\)

1.4. **Regular representation.** Let \(G\) be a finite group. Denote by \(\mathbb{C}[G]\) the space of all complex valued functions on \(G\). Clearly, \(\dim \mathbb{C}[G] = |G|\). The vector space \(\mathbb{C}[G]\) has a structure of inner product space with the inner product

\[
(f | f') = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)}
\]

for \(f, f' \in \mathbb{C}[G]\).

For \(g \in G\) and \(f \in \mathbb{C}[G]\) define the function \(R(g)f\) by \((R(g)f)(h) = f(gh)\) for any \(h \in G\). Clearly, \(R(g) : f \mapsto R(g)f\) is a linear map on \(\mathbb{C}[G]\).

Moreover, for \(g, h \in G\), we have

\[
(R(gh)f)(k) = f(kgh) = (R(h)f)(kg) = (R(g)R(h)f)(k)
\]

for any \(k \in K\). Therefore \(R(gh) = R(g)R(h)\). Clearly, \(R(1) = I\). It follows that \((R, \mathbb{C}[G])\) is a representation of \(G\). We call it the **(right) regular representation** of \(G\).

1.4.1. **Lemma.** The right regular representation is unitary.

**Proof.** Clearly, for \(g \in G\), we have

\[
(R(g)f | R(g)f') = \frac{1}{|G|} \sum_{h \in G} f(hg)\overline{f'(hg)} = \frac{1}{|G|} \sum_{h \in G} f(h)\overline{f'(h)} = (f | f')
\]

for any \(f, f' \in \mathbb{C}[G]\). Therefore, \(R(g), g \in G\), are unitary operators. \(\square\)

The following property of regular representation is critical.

1.4.2. **Lemma.** Let \(g \in G\), \(g \neq 1\). Then \(R(g) \neq I\).
Proof. Denote by $\delta_h$ the function on $G$ which is 1 at point $h \in G$ and zero everywhere else. Then we have

$$(R(g)\delta_1)(h) = \delta_1(hg) = \delta_{g^{-1}}(h)$$

for any $h \in G$, i.e., $R(g)\delta_1 = \delta_{g^{-1}} \neq \delta_1$. \hfill \Box

Since $R$ is a direct sum of irreducible representations of $G$, this result has a following consequence.

1.4.3. Theorem. Let $g \in G$, $g \neq 1$. Then there exists an irreducible representation $\pi$ of $G$ such that $\pi(g) \neq I$.

In other words, irreducible representations of $G$ separate points in $G$.

1.5. Abelian finite groups. Let $G$ be a finite group. Let $\pi$ be an one-dimensional representation of $G$. Then $\pi(g) = \lambda(g)I$, where $\lambda : G \rightarrow \mathbb{C}^*$ is group homomorphism of $G$ into the multiplicative group of complex numbers different than zero. This implies that $g \mapsto |\lambda(g)|$ is a homomorphism of $G$ into the multiplicative group of positive real numbers $\mathbb{R}^*$. Since 1 is the only element of that group of finite order, we conclude that $|\lambda(g)| = 1$, i.e., $\lambda$ is a homomorphism of $G$ into the group of complex numbers of absolute value equal to 1. We call such homomorphisms the characters of $G$.

Assume that $G$ is abelian finite group. Let $(\pi, V)$ be an irreducible representation of $G$. Let $g \in G$. Then

$$\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$$

for all $h \in G$. Therefore, by Schur Lemma, we see that $\pi(g) = \lambda(g)I$ for some complex number $\lambda(g) \neq 0$. By the above discussion, $\lambda$ is a character of $G$. This in turn implies that $\dim \pi = 1$.

1.5.1. Proposition. Let $G$ be a finite group. Then the following conditions are equivalent.

(i) $G$ is abelian;

(ii) all irreducible representations of $G$ are one-dimensional.

Proof. We already proved that (i) implies (ii).

Assume that all irreducible representations are one-dimensional. Let $g, h \in G$. Consider the element $a = ghg^{-1}h^{-1}$. Let $\pi$ be an irreducible representation of $G$. Then $\pi$ is one-dimensional and

$$\pi(a) = \pi(ghg^{-1}h^{-1}) = \pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = I$$

since $\pi(g)$ and $\pi(h)$ commute. By 1.4.3, this implies that $a = 1$, i.e., $ghg^{-1}h^{-1} = 1$. It follows that $gh = hg$ for all $g, h \in G$, i.e., $G$ is abelian. \hfill \Box

Hence, all irreducible representations of an abelian finite group are characters. Let $\phi$ and $\psi$ be two characters of $G$. Then we have

$$\phi(g)(\phi \mid \psi) = \frac{1}{|G|} \sum_{h \in G} \phi(gh)\overline{\psi(h)} = \frac{1}{|G|} \sum_{h \in G} \phi(h)\overline{\psi(g^{-1}h)} = \psi(g)(\phi \mid \psi)$$

for any $g \in G$. Hence, if $\phi$ and $\psi$ are different, they are orthogonal to each other. Moreover, for a character $\phi$ we have

$$\|\phi\|^2 = (\phi \mid \phi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\phi(g)} = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\phi(g)} = 1.$$
Hence, the characters form an orthonormal family of functions in $C[G]$. Moreover, we have the following result.


Proof. Since irreducible representations of $G$ are characters, $R$ is an direct sum of characters. This implies that there is a basis $e_i$, $1 \leq i \leq |G|$, and characters $\phi_i$, $1 \leq i \leq |G|$, such that $R(g)e_i = \phi_i(g)e_i$ for any $g \in G$. This in turn implies that

$$e_i(g) = (R(g)e_i)(1) = \phi_i(g)e_i(1)$$

for all $g \in G$. Since $e_i$ is a nonzero vector, we must have $e_i(1) \neq 0$. Hence $e_i$ is proportional to $\phi_i$. Therefore, $C[G]$ is spanned by characters. \qed

Let $\hat{G}$ be the set of all characters of $G$. Let $\phi, \psi$ be two characters of $G$. Define their product as $(\phi \cdot \psi)(g) = \phi(g)\psi(g)$ for all $g \in G$. This defines a binary operation on $\hat{G}$. It is easy to check that $\hat{G}$ is an abelian group with this operation. By the above result, $\hat{G}$ is finite and $|\hat{G}| = \dim C[G] = |G|$. We call $\hat{G}$ the dual group of $G$.

Applying the above discussion twice, we get $|\hat{\hat{G}}| = |\hat{G}| = |G|$.

Let $g \in G$. Then the map $\phi \mapsto \phi(g)$ is a character of $\hat{G}$. This defines a map $\alpha$ from $G$ into $\hat{\hat{G}}$. Moreover,

$$\alpha(gh)(\phi) = \phi(gh) = \phi(g)\phi(h) = \alpha(g)(\phi)\alpha(h)(\phi) = (\alpha(g) \cdot \alpha(h))(\phi)$$

for all $\phi \in \hat{G}$, i.e., $\alpha : G \rightarrow \hat{\hat{G}}$ is a group morphism.

Assume that $\alpha(g) = 1$. Then $\alpha(g)(\phi) = \phi(g) = 1$ for all $\phi \in \hat{G}$. By 1.4.3, it follows that $g = 1$. Therefore, $\alpha$ is an injective morphism. Hence, $\alpha : G \rightarrow \hat{\hat{G}}$ is a group isomorphism.

1.5.3. Theorem. Let $G$ be an abelian finite group and $\hat{G}$ its dual group. Then

(i) $|\hat{G}| = |G|$;
(ii) $\alpha : G \rightarrow \hat{\hat{G}}$ is an isomorphism.

This is a special case of Pontryagin duality.

Since characters form an orthonormal basis of $C[G]$, any function $f$ in $C[G]$ can be written as

$$f = \sum_{\phi \in \hat{G}} (f | \phi)\phi.$$

By Bessel equality, we have

$$\|f\|^2 = \sum_{\phi \in \hat{G}} |(f | \phi)|^2.$$

We define the Fourier transform $\mathcal{F}f$ of $f$ as the function on $\hat{G}$ given by

$$(\mathcal{F}f)(\phi) = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\phi(g)}, \quad \phi \in \hat{G}.$$

Therefore, the inverse Fourier transform is given by

$$f(g) = \sum_{\phi \in \hat{G}} (\mathcal{F}f)(\phi)\phi(g), \quad g \in G.$$
The above equality then implies that

$$
\|f\|^2 = \sum_{\phi \in \hat{G}} |(Ff)(\phi)|^2.
$$

This is a special case of Plancherel theorem.

1.6. **Unitarity.** Let $(\pi, V)$ be a finite-dimensional representation of $G$. Let $\langle \cdot | \cdot \rangle$ be an inner product on $V$.

Put

$$
(u | v) = \frac{1}{|G|} \sum_{g \in G} \langle \pi(g)u | \pi(g)v \rangle.
$$

Clearly, $(u, v) \mapsto (u | v)$ is a linear in first and antilinear in the second variable. Moreover, we have $(u | v) = (v | u)$. In addition,

$$
(v | v) = \frac{1}{|G|} \sum_{g \in G} \langle \pi(g) | \pi(g)v \rangle \geq 0
$$

for any $v \in V$. If $(v | v) = 0$, we have $\langle \pi(g)v | \pi(g)v \rangle = 0$ for all $g \in G$. In particular $(v | v) = 0$, and $v = 0$. Hence, $(\cdot | \cdot)$ is an inner product on $V$.

1.6.1. **Lemma.** Inner product $(\cdot | \cdot)$ is $G$-invariant.

**Proof.** Let $g \in G$. Then we have

$$
(\pi(g)u | \pi(g)v) = \frac{1}{|G|} \sum_{h \in G} \langle \pi(hg)u | \pi(hg)v \rangle = \frac{1}{|G|} \sum_{h \in G} \langle \pi(h)u | \pi(h)v \rangle = (u | v).
$$

Therefore, there exists an inner product on $V$ such that $(\pi, V)$ is a unitary representation.

1.7. **Orthogonality relations.** Let $(\nu, U)$ and $(\pi, V)$ be two irreducible representations of $G$. Let $A : U \rightarrow V$ be a linear map. Define

$$
B = \frac{1}{|G|} \sum_{g \in G} \pi(g)A\nu(g^{-1}).
$$

Then, $B$ is also a linear map from $U$ into $V$.

Let $g \in G$. Then

$$
\pi(g)B = \frac{1}{|G|} \sum_{h \in G} \pi(gh)A\nu(h^{-1}) = \frac{1}{|G|} \sum_{h \in G} \pi(h)A\nu(h^{-1}g) = B\nu(g).
$$

Hence, it follows that $B \in \text{Hom}_G(U, V)$. If $\nu$ and $\pi$ are not equivalent, by Schur Lemma, we have $B = 0$.

1.7.1. **Lemma.** Let $(\nu, U)$ and $(\pi, V)$ be two inequivalent irreducible representations of $G$. Then

$$
\frac{1}{|G|} \sum_{g \in G} \pi(g)A\nu(g^{-1}) = 0
$$

for any linear operator $A : U \rightarrow V$. 

Consider now an irreducible representation $(\pi, V)$ and a linear map $A : V \to V$. Let
\[ B = \frac{1}{|G|} \sum_{g \in G} \pi(g)A\pi(g^{-1}). \]
Then $B \in \text{Hom}_G(V, V)$. By Schur Lemma, we conclude that $B = \lambda I$ for some $\lambda \in \mathbb{C}$.

Moreover, we have
\[ \text{tr} B = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\pi(g)A\pi(g^{-1})) = \frac{1}{|G|} \sum_{g \in G} \text{tr} A = \text{tr} A. \]
This implies the following result.

1.7.2. **Lemma.** Let $(\pi, V)$ be an irreducible representation of $G$. Then
\[ \frac{1}{|G|} \sum_{g \in G} \pi(g)A\pi(g^{-1}) = \frac{\text{tr} A}{\dim \pi} I \]
for any linear operator $A : V \to V$.

By 1.6.1, we can assume that $U$ and $V$ are equipped with $G$-invariant inner products. Let $(e_i; 1 \leq i \leq \dim \nu)$ and $(f_j; 1 \leq j \leq \dim \pi)$, be two orthonormal bases of $U$ and $V$ respectively. Denote by $\nu(g)_{pq}$ and $\pi(g)_{rs}$ the matrix coefficients of $\nu(g)$ and $\pi(g)$ respectively. Then we first observe that
\[ \sum_{s=1}^{\dim \pi} \sum_{p=1}^{\dim \nu} \frac{1}{|G|} \sum_{g \in G} \pi(g)_{rs} A_{sp} \nu(g^{-1})_{pq} = 0, \]
where $A_{sp}$ are matrix coefficients of $A$. Since $A$ is arbitrary, we conclude that
\[ \frac{1}{|G|} \sum_{g \in G} \pi(g)_{rs} \nu(g^{-1})_{pq} = 0 \]
for all $p, q, r, s$. Clearly, since $(\nu(g^{-1})_{pq})$ is a unitary matrix, we have $\nu(g^{-1})_{pq} = \nu(g)_{qp}$ for all $p, q$. Hence, we conclude that
\[ \frac{1}{|G|} \sum_{g \in G} \pi(g)_{rs} \nu(g)_{pq} = 0 \]
for all $p, q, r, s$.

Let $(\pi, V)$ be an irreducible representation of $G$. Denote by $M(\pi)$ the vector subspace of $\mathbb{C}[G]$ spanned by matrix coefficients of $\pi$. This subspace is independent of choice of the basis of $V$. Moreover, it depends only on the equivalence class of $\pi$.

1.7.3. **Proposition.** Let $(\pi, V)$ be an irreducible representation of $G$. Then the subspace $M(\pi)$ is an invariant subspace of the regular representation $(R, \mathbb{C}[G])$.

**Proof.** Let $(e_1, e_2, \ldots, e_n)$ be a basis of $V$. Denote by $g \mapsto \pi(g)_{ij}, 1 \leq i, j \leq n$, the matrix coefficients of $\pi$ in this basis. Then $M(\pi)$ is spanned by these functions.

Let $1 \leq p, q \leq n$. Put $f(g) = \pi(g)_{pq}$ for $g \in G$. Then we have
\[ (R(g)f)(h) = f(hg) = \pi(hg)_{pq} = \sum_{s=1}^{n} \pi(h)_{ps} \pi(g)_{sq} \]
for all $h \in G$. Therefore, $R(g)f$ is a linear combination of matrix coefficients of $\pi$, i.e., $R(g)f \in M(\pi)$. It follows that $M(\pi)$ is invariant for $R(g)$. \qed
The above calculation proves the following result.

1.7.4. **Proposition.** Let $\nu$ and $\pi$ be two inequivalent irreducible representations of $G$. Then $M(\nu) \perp M(\pi)$.

Consider now an irreducible representation $(\pi, V)$. As above, we have

$$\dim \pi \sum_{s=1}^{\dim \pi} \frac{1}{[G]} \sum_{g \in G} \pi(g)_{rs} \pi(g^{-1})_{pq} = \frac{\text{tr} A}{\dim \pi} \delta_{rq}.$$  

By selecting $A$ such that $A_{kl} = 1$ for some $k \neq l$, and all other entries are 0, we get

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lq} = 0.$$  

If we select $A$ such that $A_{kk} = 1$ for some $k$, and all other entries are 0, we get

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{kq} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}.$$  

Therefore, we have

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lq} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}$$  

and

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{ql} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}$$

for all $1 \leq k, l, q, r \leq \dim \pi$. These are Schur orthogonality relations. This implies that $(\pi(g)_{ij}; 1 \leq i, j \leq \dim \pi)$ is an orthogonal basis of $M(\pi)$.

1.7.5. **Theorem.** Let $(\pi, V)$ be an irreducible representation of $G$. Then $\dim M(\pi) = (\dim \pi)^2$.

The next result describes the structure of regular representation.

1.7.6. **Theorem.** We have

$$\mathbb{C}[G] = \bigoplus_{\pi \in \hat{G}} M(\pi).$$

**Proof.** By 1.7.3, the subspaces $M(\pi)$, $\pi \in \hat{G}$, are invariant subspaces of $(R, \mathbb{C}[G])$. Therefore, their orthogonal sum $M = \bigoplus_{\pi \in \hat{G}} M(\pi)$ is an invariant subspace in $(R, \mathbb{C}[G])$.

Let $M^\perp$ be the orthogonal complement of $M$. Then $M^\perp$ is also an invariant subspace since $R$ is unitary. Assume that $M^\perp$ is different from $\{0\}$. Then it contains an irreducible representation $(\nu, U)$ of $G$ by 1.2.3. Let $(f_1, f_2, \ldots, f_m)$ be a basis of $U$. Then we have

$$\nu(g) f_i = \sum_{j=1}^{m} \pi(g)_{ji} f_j.$$

Therefore, we have

$$f_i(g) = (R(g) f_i)(1) = (\nu(g) f_i)(1) = \sum_{j=1}^{m} \nu(g)_{ji} f_j(1)$$
for all \( g \in G \). Hence, we have \( f_i \in M(\nu) \subseteq M \). Therefore, \( f_i \) is orthogonal on itself, and \( f_i = 0 \). This contradicts our choice. It follows that \( M^\perp = \{0\} \), i.e., \( M = \mathbb{C}[G] \). □

This has the following consequence.

1.7.7. Corollary. We have

\[
[G] = \sum_{\pi \in \hat{G}} (\dim(\pi))^2.
\]

1.8. Characters and central functions. Let \((\pi, V)\) be a finite-dimensional representation of \( G \). Define the function \( \text{ch}(\pi) : G \rightarrow \mathbb{C} \) by

\[
\text{ch}(\pi)(g) = \text{tr} \pi(g)
\]

for \( g \in G \). The function \( \text{ch}(\pi) \) on \( G \) is called the character of \( \pi \). The character of \( \pi \) depends only on the equivalence class of \( \pi \).

1.8.1. Example. Let \((R, \mathbb{C}[G])\) be the regular representation of \( G \). For any \( g \in G \), define the function \( \delta_g \) which is equal 1 at \( g \) and 0 everywhere else. Clearly, \((\delta_g, g \in G)\) is a basis of \( \mathbb{C}[G] \).

Let \( g \in G \). Then we have

\[
(R(g)\delta_h)(k) = \delta_h(kg) = \begin{cases} 1, & \text{if } k = hg^{-1}; \\ 0, & \text{if } k \neq hg^{-1} \end{cases} = \delta_{hg^{-1}}(k)
\]

for all \( k \in G \). Hence \( R(g)\delta_h = \delta_{hg^{-1}} \) for all \( h \in G \). It follows that the matrix of \( R(g) \) has nonzero coefficients on the diagonal if and only if \( g = 1 \). Hence we see that \( \text{tr} R(g) = 0 \) if \( g \neq 1 \) and \( \text{tr} R(1) = \dim(R) = [G] \). Therefore, we have \( \text{ch}(R) = [G] \cdot \delta_1 \).

Moreover, if \( \pi = \nu \oplus \rho \) we have

\[
\text{ch}(\pi) = \text{ch}(\nu) + \text{ch}(\rho).
\]

Hence, the character map defines a homomorphism of the Grothendieck group of \( \text{Rep}_{\text{fd}}(G) \) into functions on \( G \).

1.8.2. Theorem. (i) Let \((\pi, V)\) and \((\nu, U)\) be two irreducible representations of \( G \). If \( \pi \) is not equivalent to \( \nu \) we have \( (\text{ch}(\pi) | \text{ch}(\nu)) = 0 \).

(ii) Let \((\pi, V)\) be irreducible representation of \( G \). Then we have \( (\text{ch}(\pi) | \text{ch}(\pi)) = 1 \).

Proof. This follows immediately from Schur orthogonality relations. □

Therefore, \((\text{ch}(\pi); \pi \in \hat{G})\) is an orthonormal family of functions in \( \mathbb{C}[G] \).

Moreover we see that

\[
\dim \text{Hom}_G(U, V) = (\text{ch}(\nu) | \text{ch}(\pi))
\]

for any two finite-dimensional representations of \( G \).

Clearly, if \( g, h \in G \) we have

\[
\text{ch}(\pi)(hgh^{-1}) = \text{tr}(\pi(hgh^{-1})) = \text{tr}(\pi(h)\pi(g)\pi(h)^{-1}) = \text{tr}(\pi(g) = \text{ch}(\pi)(g).
\]

Hence, characters are constant on conjugacy classes in \( G \).

This has the following consequence.
1.8.3. Proposition. Let $(\pi, V)$ be an irreducible representation of $G$. Let $f$ be a matrix coefficient of $\pi$. Then

$$\frac{1}{|G|} \sum_{h \in G} f(hg^{-1}) = \frac{f(1)}{\dim \pi} \text{ch}(\pi)(g)$$

for any $g \in G$.

Proof. Clearly, both sides of the equality are linear forms in $f$ on the space $M(\pi)$. Therefore, it is enough to check the equality on a basis of $M(\pi)$.

By 1.6.1 we can assume that $\pi$ is unitary. Let $(e_i; 1 \leq i \leq \dim \pi)$ be an orthonormal basis of $V$. Let $g \mapsto \pi(g)_{ij}$ the matrix coefficients of $\pi$ in that basis. Then they are a basis of $M(\pi)$.

For these functions we have

$$\frac{1}{|G|} \sum_{h \in G} \pi(hg^{-1})_{ij} = \frac{1}{|G|} \sum_{h \in G} \left( \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(h)_{ik} \pi(g)_{kl} \pi(h^{-1})_{lj} \right)$$

$$= \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(g)_{kl} \left( \frac{1}{|G|} \sum_{h \in G} \pi(h)_{ik} \pi(h)_{lj} \right) = \frac{1}{\dim \pi} \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(g)_{kl} \delta_{ij} \delta_{kl}$$

$$= \frac{1}{\dim \pi} \sum_{k=1}^{\dim \pi} \pi(g)_{kk} \delta_{ij} = \frac{1}{\dim \pi} \text{ch}(\pi)(g) \delta_{ij} = \frac{1}{\dim \pi} \text{ch}(\pi)(g)\pi(1)_{ij}.$$

using Schur orthogonality relations. \qed

We say that a function $f$ on $G$ is central if it is constant on conjugacy classes in $G$. Denote by $C(G)$ the vector subspace of $\mathbb{C}[G]$ consisting of all central functions. Clearly, the dimension of $C(G)$ is equal to the number of conjugacy classes in $G$.

By 1.8.2, $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family of functions in $C(G)$.

1.8.4. Theorem. $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal basis of $C(G)$.

Proof. We already know that $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family in $C(G)$.

Let $f$ be a central function on $G$ orthogonal on all characters $\text{ch}(\pi)$, $\pi \in \hat{G}$. Let $\phi \in M(\pi)$, then we have

$$(\phi \mid f) = \frac{1}{|G|} \sum_{g \in G} \phi(g)f(g) = \frac{1}{|G|} \sum_{g \in G} \left( \frac{1}{|G|} \sum_{h \in G} \phi(g)f(h^{-1}gh) \right)$$

$$= \frac{1}{|G|} \sum_{h \in G} \left( \frac{1}{|G|} \sum_{g \in G} \phi(g)f(h^{-1}gh) \right) = \frac{1}{|G|} \sum_{g \in G} \phi(hg^{-1})f(g),$$

since $f$ is a central function. By 1.8.3, it follows that

$$(\phi \mid f) = \frac{f(1)}{\dim \pi} \frac{1}{|G|} \sum_{g \in G} \text{ch}(\pi)(g)f(g) = \frac{f(1)}{\dim \pi} \text{ch}(\pi)(f) = 0.$$

Hence $f$ is orthogonal to $M(\pi)$ for all $\pi \in \hat{G}$. By 1.7.6, it follows that $f$ is orthogonal to $\mathbb{C}[G]$. Hence $f = 0$. Therefore, $(\text{ch}(\pi); \pi \in \hat{G})$ is a maximal orthonormal family in $C(G)$, i.e., it is an orthonormal basis. \qed

Therefore, $\dim C(G)$ is equal to $\text{Card}(\hat{G})$. This implies the following result.
1.8.5. **Corollary.** \( \text{Card}(G) \) is equal to the number of conjugacy classes in \( G \).

2. **Frobenius Reciprocity**

2.1. **Restriction functor.** Let \( G \) be a a finite group. Let \( H \) be the a subgroup of \( G \). Denote by \( \text{Rep}(G) \), resp. \( \text{Rep}(H) \), the categories of representations of \( G \), resp. \( H \).

Let \((\pi, V)\) be a representation in \( \text{Rep}(G) \). Denote by \( \nu \) the restriction of function \( \pi : G \longrightarrow \text{GL}(V) \) to \( H \). Then \((\nu, V)\) is a representation in \( \text{Rep}(H) \). This representation is called the restriction of \( \pi \) to \( H \) and denoted by \( \text{Res}_H^G(\pi) \) (when there is no ambiguity we shall just write \( \text{Res}(\pi) \)).

Clearly, \( \text{Res}_H^G \) is an exact functor form the abelian category \( \text{Rep}(G) \) into the abelian category \( \text{Rep}(H) \).

2.2. **Induction functor.** Let \((\nu, U)\) be a representation of \( H \). Denote by \( V = \text{Ind}(U) \) the space of all functions \( F : G \longrightarrow U \) such that \( F(hg) = \nu(h)F(g) \) for all \( h \in H \) and \( g \in G \). Let \( F \) be the function in \( V \) and \( g \in G \). Then the function \( \rho(g)F : G \longrightarrow U \) defined by \( (\rho(g)F)(g') = F(g'g) \) for all \( g' \in G \), satisfies

\[
(\rho(g)F)(hg') = F(hg'g) = \nu(h)F(g'g) = \nu(h)(\rho(g)F)(g')
\]

for all \( h \in H \) and \( g' \in G \). Therefore \( \rho(g)F \) is a function in \( V \).

Clearly \( \rho(g) \) is a linear operator on \( V \) for any \( g \in G \). Moreover, \( \rho(1) \) is the identity on \( V \). For any \( F \) in \( V \) we have

\[
(\rho(gg')F)(g'') = F(g''gg') = (\rho(g')F)(g'') = (\rho(g)(\rho(g')F))(g'')
\]

for all \( g'', g' \in G \), i.e., we have

\[
\rho(gg')F = \rho(g)(\rho(g')F)
\]

for \( g, g' \in G \). Therefore, \( \rho(gg') = \rho(g)\rho(g') \) for any \( g, g' \in G \) and \( \rho \) is a representation of \( G \) on \( V \).

The representation \((\rho, V)\) of \( G \) is called the induced representation and denoted by \( \text{Ind}_H^G(\nu) \).

If \( H \) is the identity subgroup and \( \nu \) is the trivial representation, the corresponding induced representation is the regular representation of \( G \).

Let \((\nu, U)\) and \((\nu', U')\) be two representations of \( H \) and \( \phi \) a morphism of \( \nu \) into \( \nu' \). Let \( F \) be a function in \( \text{Ind}(U) \). Then \( \Phi(F)(g) = \phi(F(g)) \) for all \( g \in G \) is a function from \( G \) into \( U' \). Moreover, we have

\[
\Phi(F)(hg) = \phi(F(hg)) = \phi(\nu(h)F(g)) = \nu'(h)\phi(F(g)) = \nu'(h)\Phi(F)(g)
\]

for all \( h \in H \) and \( g \in G \). Hence, \( \Phi(F) \) is in \( \text{Ind}(U') \). Clearly, \( \Phi \) is a linear map from \( \text{Ind}(U) \) into \( \text{Ind}(U') \).

Moreover, we have

\[
(\rho'(g)\Phi(F))(g') = \Phi(F)(g'g) = \phi(F(g'g)) = \phi((\rho(g)F)(g')) = \Phi(\rho(g)F)(g')
\]

for all \( g' \in G \). Therefore, \( \rho'(g) \circ \Phi = \Phi \circ \rho(g) \) for all \( g \in G \), and \( \Phi \) is a morphism of \( \text{Ind}_H^G(\nu) \) into \( \text{Ind}_H^G(\nu') \). We put \( \text{Ind}_H^G(\phi) = \Phi \). It is straightforward to check that in this way \( \text{Ind}_H^G \) becomes an additive functor from \( \text{Rep}(H) \) into \( \text{Rep}(G) \).

We call \( \text{Ind}_H^G : \text{Rep}(H) \longrightarrow \text{Rep}(G) \) the induction functor.

The next result is a functorial form of Frobenius reciprocity.
2.2.1. **Theorem.** The induction functor \( \text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G) \) is a right adjoint functor of the restriction functor \( \text{Res}^G_H : \text{Rep}(G) \to \text{Rep}(H) \).

**Proof.** Let \((\nu, U)\) a representation of \( H \). Consider the induced representation \( \text{Ind}^G_H(\nu) \) of \( G \). The evaluation map \( e : \text{Ind}(U) \to U \) given by \( e(F) = F(1) \) for \( F \in \text{Ind}(U) \), satisfies

\[
 e(\rho(h)F)(1) = (\rho(h)F)(1) = F(h) = \nu(h)F(1) = \nu(h)e(F)
\]

for all \( F \in \text{Ind}(U) \), i.e., \( e \) is a morphism of representations of \( H \).

Let \((\pi, V)\) be a representation of \( G \). Let \( \Psi : V \to \text{Ind}(U) \) be a morphism of representations of \( G \). Then the composition \( e \circ \Psi \) is a morphism of \( \text{Res}^G_H(\pi) \) into \( \nu \). Denote the linear map \( \Psi \mapsto e \circ \Psi \) from \( \text{Hom}_G(\pi, \text{Ind}^G_H(\nu)) \) into \( \text{Hom}_H(\text{Res}^G_H(\pi), \nu) \) by \( A \).

Let \( \phi : V \to U \) be a morphism of representations of \( H \). Let \( v \in V \). Then we consider the function \( F_v : G \to U \) given by \( F_v(g) = \phi(\pi(g)v) \) for any \( g \in G \). First, for \( h \in H \), we have

\[
 F_v(hg) = \phi(\pi(hg)v) = \phi(\pi(h)\pi(g)v) = \nu(h)\phi(\pi(g)v) = \nu(h)F_v(g)
\]

for all \( g \in G \). Hence \( F_v \) is a function in \( \text{Ind}(U) \). Consider the map \( \Phi : V \to \text{Ind}(U) \) defined by \( \Phi(v) = F_v \). Clearly,

\[
 \Phi(v + v')(g) = F_{v + v'}(g) = \phi(\pi(g)(v + v')) = \phi(\pi(g)v) + \phi(\pi(g)v')
\]

\[
 = F_v(g) + F_{v'}(g) = \Phi(v)(g) + \Phi(v')(g)
\]

for any \( g \in G \), hence we have \( \Phi(v + v') = \Phi(v) + \Phi(v') \) for all \( v, v' \in V \). In addition,

\[
 \Phi(\alpha v)(g) = \alpha \phi(\pi(g)v) = \alpha \Phi(v)(g)
\]

for all \( g \in G \), hence we have \( \Phi(\alpha v) = \alpha \Phi(v) \) for all \( \alpha \in \mathbb{C} \) and \( v \in V \). It follows that \( \Phi \) is a linear map from \( V \) into \( \text{Ind}(U) \). Moreover, we have

\[
 \Phi(\pi(g)v)(g') = \phi(\pi(g')\pi(g)v) = \phi(\pi(g'g)v) = \Phi(v)(g'g) = (\rho(g)\Phi(v))(g')
\]

for all \( g' \in V \). Hence, we have \( \Phi(\pi(g)v) = \rho(g)\Phi(v) \) for all \( g \in G \) and \( v \in V \). Therefore, \( \Phi \) is a morphism of representations \( (\pi, V) \) and \( \text{Ind}^G_H(\nu) \) of \( G \). Denote the map \( \phi \mapsto \Phi \) from \( \text{Hom}_H(\text{Res}^G_H(\pi), \nu) \) into \( \text{Hom}_G(\pi, \text{Ind}^G_H(\nu)) \) by \( B \).

Clearly, for \( \phi \in \text{Hom}_H(\text{Res}^G_H(\pi), \nu) \), we have

\[
 ((A \circ B)(\phi))(v) = (A(\Phi))(v) = \Phi(v)(1) = F_v(1) = \phi(v)
\]

for all \( v \in V \). Therefore, \( A \circ B \) is the identity map.

In addition, for \( \Psi \in \text{Hom}_G(\pi, \text{Ind}^G_H(\nu)) \), we have

\[
 (((B \circ A)(\Psi))(v))(g) = (B(A(\Psi))(v))(g) = A(\Psi)(\pi(g)v)
\]

\[
 = (\Psi(\pi(g)v))(1) = (\rho(g)\Psi(v))(1) = \Psi(v)(g)
\]

for all \( g \in G \). Hence, we have \( ((B \circ A)(\Psi))(v) = \Psi(v) \) for all \( v \in V \), i.e., \( (B \circ A)(\Psi) = \Psi \) for all \( \Psi \) and \( B \circ A \) is also the identity map.

By Maschke’s theorem, \( \text{Rep}(H) \) is semisimple, and every short exact sequence splits. Therefore we have the following result.

2.2.2. **Theorem.** The induction functor \( \text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G) \) is exact.
2.3. **Induction in stages.** Let $K$ be a subgroup of $H$. Then we have $\text{Res}^G_K = \text{Res}^H_K \circ \text{Res}^G_H$ as functors from $\text{Rep}(G)$ into $\text{Rep}(K)$. Since induction functors are right adjoints, this immediately implies the following result which is called the *induction in stages*.

2.3.1. **Theorem.** Let $H$ be a subgroup of $G$ and $K$ a subgroup of $H$. Then the functors $\text{Ind}^G_K$ and $\text{Ind}^G_H \circ \text{Ind}^H_K$ are isomorphic.

2.4. **Frobenius Reciprocity.** Obviously, the restriction functor $\text{Res}^G_H$ maps finite-dimensional representations into finite dimensional representations. From the following result we see that the induction functor $\text{Ind}^G_H$ does the same.

2.4.1. **Proposition.** Let $(\nu, U)$ be a finite-dimensional representation of $H$. Then

$$\dim \text{Ind}^G_H(\nu) = \text{Card}(H \setminus G) \cdot \dim(\nu).$$

**Proof.** Let $C$ be a right $H$-coset in $G$. Let $g_C$ be an element in $C$. Then the functions

$$F_{C,v}(g) = \begin{cases} \nu(gg_C^{-1})v & \text{for } g \in Hg_C; \\ 0 & \text{for } g \notin Hg_C; \end{cases}$$

span $\text{Ind}(U)$. If $e_1, e_2, \ldots, e_m$ is a basis of $U$, the family $F_{C,e_i}$, $C \in H \setminus G$, $1 \leq i \leq m$, is a basis of $\text{Ind}(U)$. \hfill \Box$

Let $(\pi, V)$ be an irreducible representation of $G$ and $\nu$ an irreducible representation of $H$. Then $\text{Ind}^G_H(\nu)$ is finite-dimensional by 2.4.1 and a direct sum of irreducible representations of $G$. The multiplicity of $\pi$ in this direct sum is $\text{dim}_C \text{Hom}_G(\pi, \text{Ind}^G_H(\nu))$ by Schur Lemma. By 2.2.1, we conclude that

$$\text{dim}_C \text{Hom}_G(\pi, \text{Ind}^G_H(\nu)) = \text{dim}_C \text{Hom}_H(\text{Res}^G_H(\pi), \nu).$$

The latter expression is the multiplicity of $\nu$ in $\text{Res}^G_H(\pi)$.

This leads to the following version of Frobenius reciprocity for representations of finite groups.

2.4.2. **Theorem.** Let $\pi$ be an irreducible representation of $G$ and $\nu$ an irreducible representation of $H$. Then the multiplicity of $\pi$ in $\text{Ind}^G_H(\nu)$ is equal to the multiplicity of $\nu$ in $\text{Res}^G_H(\pi)$.

2.5. **An example.** Let $S_3$ be the symmetric group in three letters. We shall show how above results allow us to construct irreducible representations of $S_3$.

The order of $S_3$ is $3! = 6$. It contains the normal subgroup $A_3$ consisting of all even permutations which is of order 3. The quotient group $S_3/A_3$ consists of two elements.

The identity element is $(1 \ 2 \ 3)$. The other even permutations are $(2 \ 3 \ 1)$ and $(3 \ 1 \ 2)$. We have $(2 \ 1 \ 3)^2 = 1$ and

$$(2 \ 1 \ 3)(2 \ 3 \ 1)(2 \ 1 \ 3) = (3 \ 1 \ 2).$$

Hence nontrivial even permutations form a conjugacy class.

The odd permutations are $(2 \ 1 \ 3)$, $(1 \ 3 \ 2)$ and $(3 \ 2 \ 1)$. Since $(2 \ 1 \ 3)(1 \ 3 \ 2)(2 \ 1 \ 3) = (3 \ 1 \ 2)$, $(1 \ 3 \ 2)$ and $(3 \ 2 \ 1)$ are conjugate. On the other hand, $(1 \ 3 \ 2)^2 = 1$ and

$$(1 \ 3 \ 2)(2 \ 3 \ 1)(1 \ 3 \ 2) = (3 \ 2 \ 1), \text{ and } (2 \ 3 \ 1) \text{ and } (3 \ 2 \ 1) \text{ are conjugate. Therefore all odd permutations form a conjugacy class. It follows that } S_3 \text{ has three conjugacy classes. Therefore } S_3 \text{ has three irreducible representations.}$$
Clearly, two irreducible representations of $S_3$ are the trivial representation and the sign representation. Since $1^2 + 1^2 + 2^2 = 6$, by Burnside theorem, the third irreducible representation $\pi$ is two-dimensional. By 1.8.1, the character of regular representation is 6 at the identity element and 0 on all other elements. By Burnside theorem the character of $\pi$ is one half of the difference of the characters of regular representation and the direct sum of trivial and sign representation. The latter theorem the character of $\pi$ is 2 on even elements and 0 on odd elements. Therefore, the character of $\pi$ is 2 at the identity, $-1$ on nontrivial even elements and 0 at odd elements. It follows that the character of $\pi$ is supported on $A_3$.

The group $A_3$ is cyclic with three elements. It has two nontrivial one-dimensional representations. If we pick a generator $a = (231)$ of $A_3$ one character maps $a$ into $e^{i\frac{2\pi}{3}}$ and the other maps $a$ to $e^{-i\frac{2\pi}{3}}$. We call the first one $\nu$. By a direct calculation we see that $(231)a(231) = a^{-1}$. The restriction of $\pi$ to $A_3$ is a direct sum of two characters of $A_3$. Since we know that $\text{ch}(\pi)(a) = -1$ we see that it must be

$$\nu(a) + \nu(a)^{-1} = e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1.$$ 

Therefore, $\text{Res}^{S_3}_{A_3}(\pi) = \nu \oplus \nu^{-1}$.

By Frobenius reciprocity, we have

$$\dim \text{Hom}_{S_3}(\pi, \text{Ind}^{S_3}_{A_3}(\nu)) = \dim \text{Hom}_{A_3}(\text{Res}^{S_3}_{A_3}(\pi), \nu) = 1.$$ 

Hence, $\pi$ is equivalent to a subrepresentation of $\text{Ind}^{S_3}_{A_3}(\nu)$. Since their dimensions are equal, we have $\pi \cong \text{Ind}^{S_3}_{A_3}(\nu)$. Analogously, we prove that $\pi \cong \text{Ind}^{S_3}_{A_3}(\nu^{-1})$.

Therefore we proved that the dual of $S_3$ consists of the classes of the trivial representation, sign representation and the induced representation $\text{Ind}^{S_3}_{A_3}(\nu) \cong \text{Ind}^{S_3}_{A_3}(\nu^{-1})$.

2.6. **Characters of induced representations.** Let $(\nu, U)$ be a finite-dimensional representation of $H$. Let $(e_i; 1 \leq i \leq n)$ be a basis of $U$. In the proof of 2.4.1, we constructed a basis $(F_{C,i}; C \in H \backslash G, 1 \leq i \leq n)$ of $\text{Ind}(U)$. Let $C \in H \backslash G$ and $1 \leq i \leq n$. Let $g \in G$. Then

$$(\rho(g)F_{C,i})(g') = F_{C,i}(g'g)$$

for all $g' \in G$, i.e., $\rho(g)F_{C,i}$ is supported on the coset $D = C \cdot g^{-1}$. Therefore, it is a linear combination of $F_{D,j}, 1 \leq j \leq n$, i.e.,

$$\rho(g)F_{C,i} = \sum_{j=1}^{n} c_j F_{D,j}.$$ 

Hence, $\rho(g)F_{C,i}$ is a linear combination of $F_{C,j}, 1 \leq j \leq n$, if and only if $D = C$, i.e., $g_C$ and $gcg_C^{-1}$ are in the same $H$-coset. This implies that $gcg = h g_C$ for some $h \in H$, i.e., $g_C^{-1} g g_C^{-1} = h \in H$. Conversely, if $g_C^{-1} g g_C^{-1} \in H$ for some $C$, we have

$$C = H g_C = H g_C g = C \cdot g$$

and $g_C$ and $gcg$ are in the same $H$-coset. Moreover, we have

$$(\rho(g)F_{C,i})(g_C) = F_{C,i}(g_C g) = F_{C,i}(h g_C) = \nu(h) F_{C,i}(g_C)$$

$$= \nu(h) e_i = \sum_{j=1}^{n} \nu(h)_{ji} e_j = \sum_{j=1}^{n} \nu(h)_{ji} F_{C,j}(g_C).$$
This in turn implies that
\[ \rho(g)_{F_{C,i}} = \sum_{j=1}^{n} \nu(h)_{j} F_{C,j} \]
if \( C \cdot g^{-1} = C \). Therefore, the matrix of \( \rho(g) \) has a nonzero diagonal entry in the basis \( \{ F_{C,i}, C \in H \setminus G, 1 \leq i \leq n \} \), only if \( C = C \cdot g \) and then these entries are \( \nu(h)_{jj}, 1 \leq j \leq n \). This implies that
\[
\text{ch} \left( \text{Ind}_{H}^{G}(\nu) \right)(g) = \sum_{C \cdot g = C} \text{ch}(\nu)(g_{C}g_{C}^{-1}) = \frac{1}{|H|} \sum_{h \in H} \sum_{g_{C}g_{C}^{-1} \in H} \text{ch}(\nu)(h_{C}g_{C}g_{C}^{-1}h^{-1}) = \frac{1}{|H|} \sum_{g'g^{-1} \in H} \text{ch}(\nu)(g'g'^{-1}).
\]

We extend the character of \( \nu \) to a function \( \chi_{\nu} \) on \( G \) which vanishes outside \( H \). Then we get the following result.

2.6.1. Theorem. The character of induced representation \( \text{Ind}_{H}^{G}(\nu) \) is equal to
\[
\text{ch}(\text{Ind}_{H}^{G}(\nu))(g) = \frac{1}{|H|} \sum_{g' \in G} \chi_{\nu}(g'g'^{-1}).
\]

Therefore the character of the induced representation is proportional to the average of the function \( \chi_{\nu} \) on the equivalence classes in \( G \).

In particular we have the following result.

2.6.2. Corollary. The character of \( \text{Ind}_{H}^{G}(\nu) \) is supported in the union of conjugacy classes in \( G \) which intersect \( H \).

The result is particularly simple if \( H \) is a normal subgroup of \( G \).

2.6.3. Corollary. Let \( H \) be a normal subgroup of \( G \). Then:
\begin{enumerate}
\item the support of the character of \( \text{Ind}_{H}^{G}(\nu) \) is in \( H \);
\item we have
\[
\text{ch}(\text{Ind}_{H}^{G}(\nu))(h) = \frac{1}{|H|} \sum_{g \in G} \text{ch}(\nu)(ghg^{-1})
\]
for any \( h \in H \).
\end{enumerate}

2.7. An example. Consider again the representation \( \pi \cong \text{Ind}_{A_{3}}^{S_{3}}(\nu) \). By the above formula, its character vanishes outside of \( A_{3} \) and is equal to
\[
\text{ch}(\pi)(h) = \frac{1}{3} \sum_{g \in S_{3}} \nu(ghg^{-1})
\]
for \( h \in A_{3} \). If \( h = 1 \), we see that
\[
\text{ch}(\pi)(1) = \frac{6}{3} = 2.
\]
If \( h = a \), we have \( gag^{-1} = a \) for \( g \in A_{3} \). If \( g \) is not in \( A_{3} \), it is in the other \( A_{3} \)-coset. Therefore, it is in the coset represented by \( (2 1 3) \). By the calculation done before,
$gag^{-1} = a^{-1}$ for $g \notin A_3$. Therefore, we have

$$\text{ch}(\pi)(a) = \frac{1}{3} \sum_{g \in S_3} \nu(gag^{-1}) = \nu(a) + \nu(a^{-1}) = -1.$$ 

This agrees with the calculation of the character of $\pi$ done before.

**2.8. Characters and Frobenius reciprocity.** Now we are going to give a proof of 2.4.2 based on character formula for the induced representation and the orthogonality relations.

We denote by $(\cdot \mid \cdot)_G$ the inner product on $\mathbb{C}[G]$ and by $(\cdot \mid \cdot)_H$ the inner product on $\mathbb{C}[H]$. Let $\pi$ be a finite-dimensional representation of $G$ and $\nu$ a finite-dimensional representation of $H$. Then we have

$$\begin{aligned}
(\text{ch}(\text{Ind}_G^H(\nu)) \mid \text{ch}(\pi))_G &= \frac{1}{|G|} \sum_{g \in G} \text{ch}(\text{Ind}_G^H(\nu))(g) \overline{\text{ch}(\pi)(g)} \\
&= \frac{1}{|G||H|} \sum_{g \in G} \sum_{g' \in G} \chi_\nu(g' gg'^{-1}) \overline{\text{ch}(\pi)(g)} \\
&= \frac{1}{|H|} \sum_{h \in H} \frac{1}{|G|} \sum_{g \in G} \chi_\nu(g) \overline{\text{ch}(\pi)(h)} \\
&= (\text{ch}(\nu) \mid \text{ch}(\text{Res}_H^G(\pi)))_H.
\end{aligned}$$