1. Haar measure on compact groups

1.1. Compact groups. Let $G$ be a group. We say that $G$ is a topological group if $G$ is equipped with hausdorff topology such that the multiplication $(g, h) \mapsto gh$ from the product space $G \times G$ into $G$ and the inversion $g \mapsto g^{-1}$ from $G$ into $G$ are continuous functions.

Let $G$ and $H$ be two topological groups. A morphism of topological groups $\varphi : G \to H$ is a group homomorphism which is also continuous.

Topological groups and morphisms of topological groups for the category of topological groups.

A topological group $G$ is compact, if $G$ is a compact space.

We shall need the following fact. Let $G$ be a topological group. We say that a function $\phi : G \to \mathbb{C}$ is uniformly continuous on $G$ if for any $\epsilon > 0$ there exists an open neighborhood $U$ of 1 such that $|\phi(g) - \phi(g')| < \epsilon$ for any $g, h \in G$ such that $gh^{-1} \in U$.

1.1.1. Lemma. Let $G$ be a compact group. Let $\phi$ be a continuous function on $G$. Then $\phi$ is uniformly continuous on $G$.

Proof. Let $\epsilon > 0$. Let consider the set $A = \{(g, g') \in G \times G \mid |\phi(g) - \phi(g')| < \epsilon\}$. Then $A$ is an open set in $G \times G$. Let $U$ be an open neighborhood of 1 in $G$ and $B_U = \{(g, g') \in G \times G \mid g'g^{-1} \in U\}$. Since the function $(g, g') \mapsto g'g^{-1}$ is continuous on $G \times G$ the set $B_U$ is open. It is enough to show that there exists an open neighborhood $V$ of 1 in $G$ such that $B_V \subset A$.

Clearly, $B_U$ are open sets containing the diagonal $\Delta$ in $G \times G$. Moreover, under the homomorphism $\kappa$ of $G \times G$ given by $\kappa(g, g') = (g, g'g^{-1})$, $g, g' \in G$, the sets $B_U$ correspond to the sets $G \times U$. In addition, the diagonal $\Delta$ corresponds to $G \times \{1\}$. Assume that the open set $O$ corresponds to $A$.

By the definition of product topology, for any $g \in G$ there exist neighborhoods $U_g$ of 1 and $V_g$ of $g$ such that $V_g \times U_g$ is a neighborhood of $(g, 1)$ contained in $O$. Clearly, $(V_g, g \in G)$ is an open cover of $G$. Since $G$ is compact, there exists a finite subcovering $(V_{g_i}; 1 \leq i \leq n)$ of $G$. Put $U = \bigcap_{i=1}^{n} U_{g_i}$. Then $U$ is an open neighborhood of 1 in $G$. Moreover, $G \times U$ is an open set in $G \times G$ contained in $O$. Therefore $B_U \subset A$.

1.2. A compactness criterion. Let $X$ be a compact space. Denote by $C(X)$ the space of all complex valued continuous functions on $X$. Let $\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. Then $f \mapsto \|f\|$ is a norm on $C(X)$, $C(X)$ is a Banach space.

Let $S$ be a subset of $C(X)$.

We say that $S$ is equicontinuous if for any $\epsilon > 0$ and $x \in X$ there exists a neighborhood $U$ of $x$ such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in S$. 

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We say that $\mathcal{S}$ is pointwise bounded if for any $x \in X$ there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{S}$.

The aim of this section is to establish the following theorem.

1.2.1. Theorem (Arzelà-Ascoli). Let $\mathcal{S}$ be a pointwise bounded, equicontinuous subset of $C(X)$. Then the closure of $\mathcal{S}$ is a compact subset of $C(X)$.

Proof. We first prove that $\mathcal{S}$ is bounded in $C(X)$. Let $\epsilon > 0$. Since $\mathcal{S}$ is equicontinuous, for any $x \in X$, there exists an open neighborhood $U_x$ of $x$ such that $y \in U_x$ implies that $|f(y) - f(x)| < \epsilon$ for all $f \in \mathcal{S}$. Since $X$ is compact, there exists a finite set of points $x_1, x_2, \ldots, x_n \in X$ such that $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover $X$.

Since $\mathcal{S}$ is pointwise bounded, there exists $M \geq 2\epsilon$ such that $|f(x_i)| \leq \frac{M}{2}$ for all $1 \leq i \leq n$ and all $f \in \mathcal{S}$. Let $x \in X$. Then $x \in U_{x_i}$ for some $1 \leq i \leq n$. Therefore, we have

$$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < \frac{M}{2} + \epsilon \leq M$$

for all $f \in \mathcal{S}$. It follows that $\|f\| \leq M$ for all $f \in \mathcal{S}$. Hence $\mathcal{S}$ is contained in a closed ball of radius $M$ centered at 0 in $C(X)$.

Now we prove that $\mathcal{S}$ is contained in a finite family of balls of fixed small radius centered in elements of $\mathcal{S}$. We keep the choices from the first part of the proof. Let $D = \{z \in \mathbb{C} \mid |z| \leq M\}$. Then $D$ is compact. Consider the compact set $D^n$. It has natural metric given by $d(z, y) = \max_{1 \leq i \leq n} |z_i - y_i|$. There exist points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $D^n$ such that the balls $B_i = \{\beta \in D^n \mid d(\alpha_i, \beta) < \epsilon\}$ cover $D^n$.

Denote by $\Phi$ the map from $\mathcal{S}$ into $D^n$ given by $f \mapsto (f(x_1), f(x_2), \ldots, f(x_n))$. Then we can find a subfamily of the above cover of $D^n$ consisting of balls intersecting $\Phi(\mathcal{S})$. After a relabeling, we can assume that these balls are $B_i$ for $1 \leq i \leq k$. Let $f_1, f_2, \ldots, f_k$ be functions in $\mathcal{S}$ such that $\Phi(f_i)$ is in the ball $B_i$ for any $1 \leq i \leq k$.

Denote by $C_i$ the open ball of radius $2\epsilon$ centered in $\Phi(f_i)$. Let $\beta \in B_i$. Then we have $d(\beta, \alpha_i) < \epsilon$ and $d(\Phi(f_i), \alpha_i) < \epsilon$. Hence, we have $d(\beta, \Phi(f_i)) < 2\epsilon$, i.e., $B_i \subset C_i$. It follows that $\Phi(\mathcal{S})$ is contained in the union of $C_1, C_2, \ldots, C_k$.

Differently put, for any function $f \in \mathcal{S}$, there exists $1 \leq i \leq k$ such that $|f(x_j) - f_i(x_j)| < 2\epsilon$ for all $1 \leq j \leq n$.

Let $x \in X$. Then $x \in U_{x_j}$ for some $1 \leq j \leq n$. Hence, we have

$$|f(x) - f_i(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| < 4\epsilon,$$

i.e., $\|f - f_i\| < 4\epsilon$.

Now we can prove the compactness of the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$. Assume that $\bar{\mathcal{S}}$ is not compact. Then there exists an open cover $\mathcal{U}$ of $\bar{\mathcal{S}}$ which doesn’t contain a finite subcover. By the above remark, $\bar{\mathcal{S}}$ can be covered by finitely many closed balls $\{f \in C(X) \mid \|f - f_i\| \leq 1\}$ with $f_i \in \mathcal{S}$. Therefore, there exists a set $K_1$ which is the intersection of $\bar{\mathcal{S}}$ with one of the closed balls and which is not covered by a finite subcover of $\mathcal{U}$. By induction, we can construct a decreasing family $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ of closed subsets of $\bar{\mathcal{S}}$ which are contained in closed balls of radius $\frac{1}{n}$ centered in some point of $\mathcal{S}$, such that none of $K_n$ is covered by a finite subcover of $\mathcal{U}$.

Let $(F_n; n \in \mathbb{N})$ be a sequence of functions such that $F_n \in K_n$ for all $n \in \mathbb{N}$. Then $F_p, F_q \in K_n$ for all $p, q$ greater than $n$. Since $K_n$ are contained in closed balls of radius $\frac{1}{n}$, $\|F_p - F_q\| \leq \frac{2}{n}$ for all $p, q$ greater than $n$. Hence, $(F_n)$ is a Cauchy sequence in $C(X)$. Therefore, it converges to a function $F \in C(X)$. This function is in $\bar{\mathcal{S}}$ and therefore in one element $V$ of the open cover $\mathcal{U}$. Therefore,
for sufficiently large \( n \), there exists a closed ball of radius \( \frac{2}{n} \) centered in \( F \) which is contained in \( V \). Since \( F \) is also in \( K_n \), we see that \( K_n \) is in \( V \). This clearly contradicts our construction of \( K_n \). It follows that \( \bar{S} \) must be compact. \( \square \)

1.3. **Haar measure on compact groups.** Let \( \mathcal{C}_R(G) \) be the space of real valued functions on \( G \). For any function \( f \in \mathcal{C}_R(G) \) we define the maximum \( M(f) = \max_{g \in G} f(g) \) and minimum \( m(f) = \min_{g \in G} f(g) \). Moreover, we denote by \( V(f) = M(f) - m(f) \) the variation of \( f \).

Clearly, the function \( f \) is constant on \( G \) if and only if \( V(f) = 0 \).

Let \( f, f' \in \mathcal{C}_R(G) \) be two functions such that \( \|f - f'\| < \epsilon \). Then

\[
f(g) - \epsilon < f'(g) < f(g) + \epsilon
\]

for all \( g \in G \). This implies that

\[
m(f) - \epsilon < f'(g) < M(f) + \epsilon
\]

for all \( g \in G \), and

\[
m(f) - \epsilon < m(f') < M(f') < M(f) + \epsilon.
\]

Hence

\[
V(f') = M(f') - m(f') < M(f) - m(f) + 2\epsilon = V(f) + 2\epsilon,
\]

i.e., \( V(f') - V(f) < 2\epsilon \). By symmetry, we also have \( V(f) - V(f') < 2\epsilon \). It follows that \( |V(f) - V(f')| < 2\epsilon \).

Therefore, we have the following result.

1.3.1. **Lemma.** The variation \( V \) is a continuous function on \( \mathcal{C}_R(G) \).

Let \( f \in \mathcal{C}_R(G) \) and \( a = (a_1, a_2, \ldots, a_n) \) a finite sequence of points in \( G \). We define the (right) mean value \( \mu(f, a) \) of \( f \) with respect to \( a \) as

\[
\mu(f, a)(g) = \frac{1}{n} \sum_{i=1}^{n} f(ga_i)
\]

for all \( g \in G \). Clearly, \( \mu(f, a) \) is a continuous real function on \( G \).

If \( f \) is a constant function, \( \mu(f, a) = f \).

Clearly, mean value \( f \mapsto \mu(f, a) \) is a linear map. Moreover, we have the following result.

1.3.2. **Lemma.** (i) The linear map \( f \mapsto \mu(f, a) \) is continuous. More precisely, we have

\[
\|\mu(f, a)\| \leq \|f\|
\]

for any \( f \in \mathcal{C}_R(G) \);

(ii) \( M(\mu(f, a)) \leq M(f) \)

for any \( f \in \mathcal{C}_R(G) \);

(iii) \( m(\mu(f, a)) \geq m(f) \)

for any \( f \in \mathcal{C}_R(G) \);

(iv) \( V(\mu(f, a)) \leq V(f) \)

for any \( f \in \mathcal{C}_R(G) \).
Proof. (i) Clearly, we have
\[ \|\mu(f,a)\| = \max_{g \in G} |\mu(f,a)| \leq \frac{1}{n} \sum_{i=1}^{n} \max_{g \in G} |f(ga_i)| = \|f\|. \]

(ii) We have
\[ M(\mu(f,a)) = \frac{1}{n} \max_{g \in G} \left( \sum_{i=1}^{n} f(ga_i) \right) \leq \frac{1}{n} \sum_{i=1}^{n} \max_{g \in G} f(ga_i) = M(f). \]

(iii) We have
\[ m(\mu(f,a)) = \frac{1}{n} \min_{g \in G} \left( \sum_{i=1}^{n} f(ga_i) \right) \geq \frac{1}{n} \sum_{i=1}^{n} \min_{g \in G} f(ga_i) = m(f). \]

(iv) By (ii) and (iii), we have
\[ V(\mu(f,a)) = M(\mu(f,a)) - m(\mu(f,a)) \leq M(f) - m(f) = V(f). \]

Denote by \( \mathcal{M}_f \) the set of mean values of \( f \) for all finite sequences in \( G \).

1.3.3. Lemma. The set of functions \( \mathcal{M}_f \) is uniformly bounded and equicontinuous.

Proof. By 1.3.2 (ii) and (iii), it follows that
\[ m(f) \leq m(\mu(f,a)) \leq \mu(f,a)(g) \leq M(\mu(f,a)) \leq M(f). \]
This implies that \( \mathcal{M}_f \) is uniformly bounded.

Now we want to prove that \( \mathcal{M}_f \) is equicontinuous. First, by 1.1.1, the function \( f \) is uniformly continuous. Hence, for any \( \epsilon > 0 \), there exists an open neighborhood \( U \) of 1 in \( G \) such that \( |f(g) - f(h)| < \epsilon \) if \( gh^{-1} \in U \). Since, this implies that \( (ga_i)(ha_i)^{-1} = gh^{-1} \in U \) for any \( 1 \leq i \leq n \), we see that
\[ |\mu(f,a)(g) - \mu(f,a)(h)| = \frac{1}{n} \left| \sum_{i=1}^{n} (f(ga_i) - f(ha_i)) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |f(ga_i) - f(ha_i)| < \epsilon \]
for \( g \in hU \). Hence, the family \( \mathcal{M}_f \) is equicontinuous. \( \square \)

By 1.2.1, we have the following consequence.

1.3.4. Lemma. The set \( \mathcal{M}_f \) of all right mean values of \( f \) has compact closure in \( C(G) \).

We need another result on variation of mean value functions. Clearly, if \( f \) is a constant function \( \mu(f,a) = f \) for any \( a \).

1.3.5. Lemma. Let \( f \) be a function in \( C(G) \). Assume that \( f \) is not a constant. Then there exists \( a \) such that \( V(\mu(f,a)) < V(f) \).

Proof. Since \( f \) is not constant, we have \( m(f) < M(f) \). Let \( C \) be such that \( m(f) < C < M(f) \). Then there exists an open set \( V \) in \( G \) such that \( f(g) \leq C \) for all \( g \in V \). Since the right translates of \( V \) cover \( G \), by compactness of \( G \) we can find...
\( a = (a_1, a_2, \cdots, a_n) \) such that \((Va_i^{-1}, 1 \leq i \leq n)\) is an open cover of \( G \). For any \( g \in Va_i^{-1} \) we have \( ga_i \in V \) and \( f(ga_i) \leq C \). Hence, we have

\[
\mu(f, a)(g) = \frac{1}{n} \sum_{j=1}^{n} f(ga_j) = \frac{1}{n} \left( f(ga_i) + \sum_{j \neq i} f(ga_j) \right) \leq \frac{1}{n} (C + (n - 1)M(f)) < M(f).
\]

On the other hand, by 1.3.2.(iii) we know that \( m(\mu, a) \geq m(f) \) for any \( a \). Hence we have

\[
V(\mu(f, a)) = M(\mu(f, a)) - m(\mu(f, a)) < M(f) - m(f) = V(f).
\]

\[\square\]

Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_m) \) be two finite sequences in \( G \). We define \( a \cdot b = (a_ib_j; 1 \leq i \leq n, 1 \leq j \leq m) \).

1.3.6. **Lemma.** We have

\[
\mu(\mu(f, b), a) = \mu(f, b \cdot a).
\]

**Proof.** We have

\[
\mu(\mu(f, b), a) = \frac{1}{m} \sum_{i=1}^{m} \mu(\mu(f, b))(ga_i) = \frac{1}{nm} \sum_{j=1}^{m} \sum_{i=1}^{n} f(ga_jb_i) = \mu(f, a \cdot b).
\]

\[\square\]

1.3.7. **Lemma.** For any \( f \in \mathcal{C}_\mathbb{R}(G) \), the closure \( \mathcal{M}_f \) contains a constant function on \( G \).

**Proof.** By 1.3.4, we know that \( \overline{\mathcal{M}_f} \) is compact. Since, by 1.3.1, the variation \( V \) is continuous on \( \mathcal{C}_\mathbb{R}(G) \), it attains its minimum \( \alpha \) at some \( \varphi \in \overline{\mathcal{M}_f} \).

Assume that \( \varphi \) is not a constant. By 1.3.5, there exists \( a \) such that \( V(\mu(\varphi, a)) < V(\varphi) \). Let \( \alpha - V(\mu(\varphi, a)) = \epsilon > 0 \).

Since \( V \) and \( \mu(\cdot, a) \) are continuous maps by 1.3.1 and 1.3.2.(i), this implies that there is \( b \) such that \( |V(\mu(\varphi, a)) - V(\mu(\varphi, b), a)| < \frac{\epsilon}{2} \). Therefore, we have

\[
V(\mu(\mu(f, b), a)) \leq V(\mu(\varphi, a)) + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2}.
\]

By 1.3.6, we have

\[
V(\mu(f, a \cdot b)) < \alpha - \frac{\epsilon}{2}
\]

contrary to our choice of \( \alpha \).

It follows that \( \varphi \) is a constant function. In addition \( \alpha = 0 \). \[\square\]

Consider now left mean values of a function \( f \in \mathcal{C}_\mathbb{R}(G) \). We define the left mean value of \( f \) with respect to \( a = (a_1, a_2, \ldots, a_n) \) as the function

\[
\nu(f, a)(g) = \frac{1}{n} \sum_{i=1}^{n} f(a_ig)
\]

for \( g \in G \). We denote \( \mathcal{N}_f \) the set of all left mean values of \( f \).

Let \( G^{opp} \) be the compact group opposite to \( G \). Then the left mean values of \( f \) on \( G \) are the right mean values of \( f \) on \( G^{opp} \).

Hence, from 1.3.7, we deduce the following result.
1.3.8. **Lemma.** For any \( f \in C_R(G) \), the closure \( \mathcal{N}_f \) contains a constant function on \( G \).

By direct calculation we get the following result.

1.3.9. **Lemma.** For any \( f \in C_R(G) \) we have
\[
\nu(\mu(f, a), b) = \mu(\nu(f, b), a)
\]
for any two finite sequences \( a \) and \( b \) in \( G \).

*Proof.* We have
\[
\nu(\mu(f, a), b)(g) = \frac{1}{m} \sum_{j=1}^{m} \mu(f, a)(b_j g) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(b_j g a_i)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \nu(f, b)(g a_i) = \mu(f, b, a)(g)
\]
for any \( g \in G \).

Putting together these results, we finally get the following.

1.3.10. **Proposition.** For any \( f \in C_R(G) \), the closure \( \mathcal{M}_f \) contains a unique function constant on \( G \).

This function is also the unique constant function in \( \mathcal{N}_f \).

*Proof.* Let \( \varphi \) and \( \psi \) be two constant functions such that \( \varphi \) is in the closure of \( \mathcal{M}_f \) and \( \psi \) is in the closure of \( \mathcal{N}_f \). For any \( \epsilon > 0 \) we have \( a \) and \( b \) such that
\[
\|\mu(f, a) - \varphi\| < \frac{\epsilon}{2} \quad \text{and} \quad \|\nu(f, b) - \psi\| < \frac{\epsilon}{2}.
\]
On the other hand, we have
\[
\|\nu(\mu(f, a), b) - \varphi\| = \|\nu(\mu(f, a), b) - \nu(\varphi, b)\|
\]
\[
= \|\nu(\mu(f, a) - \varphi, b)\| \leq \|\mu(f, a) - \varphi\| < \frac{\epsilon}{2}.
\]
In the same way, we also have
\[
\|\mu(\nu(f, b), a) - \psi\| = \|\mu(\nu(f, b), a) - \mu(\psi, a)\|
\]
\[
= \|\mu(\nu(f, b) - \psi, a)\| \leq \|\nu(f, b) - \psi\| < \frac{\epsilon}{2}.
\]
By 1.3.9, this immediately yields
\[
\|\varphi - \psi\| \leq \|\nu(\mu(f, a), b) - \varphi\| + \|\mu(\nu(f, b), a) - \psi\| < \epsilon.
\]
This implies that \( \varphi = \psi \). Therefore, any constant function in the closure of \( \mathcal{M}_f \) has to be equal to \( \psi \).

The value of the unique constant function in the closure of \( \mathcal{M}_f \) is denoted by \( \mu(f) \) and called the *mean value* of \( f \) on \( G \). In this way, we get a function \( f \mapsto \mu(f) \) on \( C_R(G) \).

1.3.11. **Lemma.** The function \( \mu \) is a continuous linear form on \( C_R(G) \).

To prove this result we need some preparation.

1.3.12. **Lemma.** Let \( f \in C_R(G) \). Then, for any \( a \) we have
\[
\mu(\mu(f, a)) = \mu(f).
\]
Proof. Let $\mu(f) = \alpha$. Let $\varphi$ be the function equal to $\alpha$ everywhere on $G$. Fix $\epsilon > 0$. Then there exists a finite sequence $b$ such that

$$\|\nu(f, b) - \varphi\| < \epsilon.$$ 

This implies that

$$\|\nu(f - \varphi, b)\| = \|\nu(f, b) - \nu(\varphi, b)\| = \|\nu(f, b) - \varphi\| < \epsilon.$$ 

This, by 1.3.2.(i), implies that

$$\|\mu(\nu(f - \varphi, b), a)\| \leq \|\nu(f - \varphi, b)\| < \epsilon$$

for any finite sequence $a$.

By 1.3.9, we have

$$\|\nu(\mu(f - \varphi, b), a)\| = \|\nu(\mu(f - \varphi, b), a)\| < \epsilon,$$

and

$$\|\nu(\mu(f, a), b) - \varphi\| = \|\nu(\mu(f, a) - \varphi, b)\| = \|\nu(\mu(f, a), b)\| < \epsilon.$$ 

Therefore, if we fix $a$, we see that $\varphi$ is in the closure of $\mathcal{N}_{\mu(f, a)}$. By 1.3.10, this proves our assertion.

Let $f$ and $f'$ be two functions in $C_0(G)$. Let $\alpha = \mu(f)$ and $\beta = \mu(f')$. Denote by $\varphi$ and $\psi$ the corresponding constant functions. Let $\epsilon > 0$.

Clearly, there exists $b$ such that

$$\|\mu(f, b) - \varphi\| < \frac{\epsilon}{2}.$$ 

This, by 1.3.2.(ii), implies that we have

$$\|\mu(\mu(f, b), a) - \varphi\| = \|\mu(\mu(f, b) - \varphi, a)\| < \frac{\epsilon}{2}.$$ 

for arbitrary $a$. By 1.3.6, this in turn implies that

$$\|\mu(f, a \cdot b) - \varphi\| < \frac{\epsilon}{2}.$$ 

On the other hand, by 1.3.12, we have $\mu(\mu(f', b)) = \mu(f) = \alpha$. Therefore, there exists a finite sequence $a$ such that

$$\|\mu(\mu(f', b), a) - \psi\| < \frac{\epsilon}{2}.$$ 

This, by 1.3.6, implies that

$$\|\mu(f', a \cdot b) - \psi\| < \frac{\epsilon}{2}.$$ 

Hence, we have

$$\|\mu(f + f', a \cdot b) - (\varphi + \psi)\| \leq \|\mu(f, a \cdot b) - \varphi\| + \|\mu(f', a \cdot b) - \psi\| < \epsilon.$$ 

Therefore, $\varphi + \psi$ is in the closure of $\mathcal{M}_{f + f'}$. Hence we have

$$\mu(f + f') = \alpha + \beta = \mu(f) + \mu(f').$$ 

It follows that $\mu$ is additive.

Let $c \in \mathbb{R}$ and $f \in C_0(G)$. Then $\mu(cf, a) = c\mu(f, a)$ for any $a$. Therefore, $\mathcal{M}_{cf} = c\mathcal{M}_f$. This immediately implies that $\mu(cf) = c\mu(f)$. Therefore $\mu$ is a linear form.

On the other hand, by 1.3.2.(ii), we have

$$\mu(f, a)(g) \leq M(f)$$
for any \(a\) and \(g \in G\). This in turn implies that \(\mu(f) \leq M(f) \leq \|f\|\). On the other hand, this implies that
\[
-\mu(f) = \mu(-f) \leq \| -f \| = \|f\|.
\]
Hence, we have \(|\mu(f)| \leq \|f\|\) for any \(f \in C_\mathbb{R}(G)\), i.e. \(\mu\) is a continuous linear form.

This completes the proof of 1.3.11.

Assume that \(f\) is a function in \(C_\mathbb{R}(G)\) such that \(f(g) \geq 0\) for all \(g \in G\). Then \(\mu(f, a)(g) \geq 0\) for any \(a\) and \(g \in G\). Hence, any function \(\phi \in \mathcal{M}_f\) satisfies \(\phi(g) \geq 0\) for all \(g \in G\). This immediately implies that \(\phi(g) \geq 0\), \(g \in G\), for any \(\phi\) in the closure of \(\mathcal{M}_f\). It follows that \(\mu(f) \geq 0\). Hence, we say that \(\mu\) is a positive linear form. By Riesz representation theorem, the linear form \(\mu : C_\mathbb{R}(G) \to \mathbb{R}\) defines a regular positive measure \(\mu\) on \(G\) such that
\[
\mu(f) = \int_G f \, d\mu.
\]
Clearly, we have
\[
\mu(G) = \int_G d\mu = \mu(1) = 1.
\]
so we say that \(\mu\) is normalized.

Denote by \(R\) (resp. \(L\)) the right regular representation (resp. left regular representation) of \(G\) on \(C(G)\) given by \((R(g)f)(h) = f(hg)\) (resp. \((L(g)f)(h) = f(g^{-1}h)\)) for any \(f \in C(G)\) and \(g, h \in G\).

1.3.13. Lemma. Let \(f \in C_\mathbb{R}(G)\) and \(g \in G\). Then
\[
\mu(R(g)f) = \mu(L(g)f) = \mu(f).
\]

Proof. Let \(a = (a_1, a_2, \cdots, a_n)\). We denote \(a \cdot g = (a_1g, a_2g, \cdots, a_ng)\) for any \(g \in G\). We have
\[
\mu(R(g)f, a)(h) = \frac{1}{n} \sum_{i=1}^{n} (R(g)f)(ha_i) = \frac{1}{n} \sum_{i=1}^{n} f(ha_i) = \mu(f, a \cdot g)(h)
\]
for any \(h \in G\), i.e.,
\[
\mu(R(g)f, a) = \mu(f, a \cdot g).
\]
This implies that \(\mathcal{M}_{R(g)f} = \mathcal{M}_f\). It follows that \(\mu(R(g)f) = \mu(f)\).

This statement for \(G^{opp}\) implies the other equality. \(\square\)

We say that the linear form \(\mu\) is biinvariant, i.e., right invariant and left invariant.

This in turn implies that the measure \(\mu\) is biinvariant, i.e., we have the following result.

1.3.14. Lemma. Let \(A\) be a measurable set in \(G\). Then \(gA\) and \(Ag\) are also measurable and
\[
\mu(gA) = \mu(Ag) = \mu(A)
\]
for any \(g \in G\).

Proof. Since \(C_\mathbb{R}(G)\) is dense in \(L^1(\mu)\), the invariance from 1.3.13 holds for any function \(f \in L^1(\mu)\). Applying it to the characteristic function of the set \(A\) implies the result. \(\square\)
A normalized biinvariant positive measure $\mu$ on $G$ is called a Haar measure on $G$.

We proved the existence part of the following result.

1.3.15. **Theorem.** Let $G$ be a compact group. Then there exists a unique Haar measure $\mu$ on $G$.

**Proof.** We constructed a Haar measure on $G$.

It remains to prove the uniqueness. Let $\nu$ be another Haar measure on $G$. Then, by left invariance, we have

$$\int_G \mu(f, a) \, d\nu = \frac{1}{n} \sum_{i=1}^{n} \int_G f(ga_i) \, d\nu(g) = \int_G f \, d\nu$$

for any $a$. Hence the integral with respect to $\nu$ is constant on $M_f$. By continuity, it is also constant on its closure. Therefore, we have

$$\int_G f \, d\nu = \mu(f) \int_G \, d\nu = \mu(f) = \int_G d\mu$$

for any $C_G(G)$. This in turn implies that $\nu = \mu$. \hfill \Box

1.3.16. **Lemma.** Let $\mu$ be the Haar measure on $G$. Let $U$ be a nonempty open set in $G$. Then $\mu(U) > 0$.

**Proof.** Since $U$ is nonempty, $(Ug; g \in G)$ is an open cover of $G$. It contains a finite subcover $(Ug_i; 1 \leq i \leq n)$. Therefore we have

$$1 = \mu(G) = \mu \left( \bigcup_{i=1}^{n} Ug_i \right) \leq \sum_{i=1}^{n} \mu(Ug_i) = \sum_{i=1}^{n} \mu(U) = n \mu(U)$$

by 1.3.14. This implies that $\mu(U) \geq \frac{1}{n}$. \hfill \Box

1.3.17. **Lemma.** Let $f$ be a continuous function on $G$. Then

$$\int_G f(g^{-1}) \, d\mu(g) = \int_G f(g) \, d\mu(g).$$

**Proof.** Clearly, it is enough to prove the statement for real-valued functions. Therefore, we can consider the linear form $\nu : f \mapsto \int_G f(g^{-1}) \, d\mu(g)$. Clearly, this a positive continuous linear form on $C_G(G)$. Moreover,

$$\nu(f) = \int_G f(h^{-1}) \, d\mu(h) = \int_G f((hg)^{-1}) \, d\mu(h)$$

$$= \int_G f(g^{-1}h^{-1}) \, d\mu(h) = \int_G (L(g)f)(h^{-1}) \, d\mu(h) = \nu(L(g)f)$$

and

$$\nu(f) = \int_G f(h^{-1}) \, d\mu(h) = \int_G f((g^{-1}h)^{-1}) \, d\mu(h)$$

$$= \int_G f(h^{-1}g) \, d\mu(h) = \int_G (R(g)f)(h^{-1}) \, d\mu(h) = \nu(R(g)f)$$

for any $g \in G$. Hence, this linear form is left and right invariant. By the uniqueness of the Haar measure we get the statement. \hfill \Box
2. Algebra of matrix coefficients

2.1. Topological vector spaces. Let $E$ be a vector space over $\mathbb{C}$. We say that $E$ is a topological vector space over $\mathbb{C}$, if it is also equipped with a topology such that the functions $(u,v) \mapsto u + v$ from $E \times E$ into $E$, and $(\alpha,u) \mapsto \alpha u$ from $\mathbb{C} \times E$ into $E$ are continuous.

A morphism $\varphi : E \rightarrow F$ of topological vector space $E$ into $F$ is a continuous linear map from $E$ to $F$.

We say that $E$ is a hausdorff topological vector space if the topology of $E$ is hausdorff.

The vector space $\mathbb{C}^n$ with its natural topology is a hausdorff topological vector space.

Let $E$ be a topological vector space and $F$ a vector subspace of $E$. Then $F$ is a topological vector space with the induced topology. Moreover, if $E$ is hausdorff, $F$ is also hausdorff.

2.1.1. Lemma. Let $E$ be a topological vector space over $\mathbb{C}$. Then the following conditions are equivalent:

(i) $E$ is hausdorff;

(ii) $\{0\}$ is a closed set in $E$.

Proof. Assume that $E$ is hausdorff. Let $v \in E$, $v \neq 0$. Then there exist open neighborhoods $U$ of 0 and $V$ of $v$ such that $U \cap V = \emptyset$. In particular, $V \subset E - \{0\}$. Hence, $E - \{0\}$ is an open set. This implies that $\{0\}$ is closed.

Assume now that $\{0\}$ is closed in $E$. Then $E - \{0\}$ is an open set. Let $u$ and $v$ be different vectors in $E$. Then $u - v \neq 0$. Since the function $(x,y) \mapsto x + y$ is continuous, there exist open neighborhoods $U$ of $u$ and $V$ of $v$ such that $U - V \subset E - \{0\}$. This in turn implies that $U \cap V = \emptyset$. □

The main result of this section is the following theorem. It states that hausdorff finite-dimensional topological vector spaces have unique topology.

2.1.2. Theorem. Let $E$ be a finite-dimensonal hausdorff topological vector space over $\mathbb{C}$. Let $v_1, v_2, \ldots, v_n$ be a basis of $E$. Then the linear map $\mathbb{C}^n \rightarrow E$ given by

$$(c_1, c_2, \ldots, c_n) \mapsto \sum_{i=1}^{n} c_i v_i$$

is an isomorphism of topological vector spaces.

Clearly, the map

$$\phi(z) = \sum_{i=1}^{n} z_i v_i,$$

for all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, is a continuous linear isomorphism of $\mathbb{C}^n$ onto $E$. Therefore, it is enough to show that that map is also open.

We start with an elementary lemma.

2.1.3. Lemma. Let $E$ and $F$ be topological vector spaces and $\phi : E \rightarrow F$ a linear map. Let $(U_i, i \in I)$ be a fundamental system of neighborhoods of $0$ in $E$. If $\phi(U_i)$, $i \in I$, are neighborhoods of $0$ in $F$, $\phi$ is an open map.

Proof. Let $U$ be an open set in $E$. For any $u \in U$ there exists $i \in I$ such that $u + U_i$ is a neighborhood of $u$ contained in $U$. Therefore, $\phi(u + U_i) = \phi(u) + \phi(U_i)$ is a
neighborhood of $\phi(u)$ contained in $\phi(U)$. Hence, $\phi(u)$ is an interior point of $\phi(U)$. This implies that $\phi(U)$ is open. □

We consider on $\mathbb{C}^n$ the standard euclidean norm $\| \cdot \|$. Let $B_1 = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the open unit ball in $\mathbb{C}^n$. Then the balls $\{\epsilon B_1 \mid 0 < \epsilon < \infty\}$ form a fundamental system of neighborhoods of 0 in $\mathbb{C}^n$. By 2.1.3, to show that the above map $\phi : \mathbb{C}^n \to E$ is open it is enough to show that $\phi(B_1)$ is a neighborhood of 0 in $E$.

Let $S = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Then, $S$ is a bounded and closed set in $\mathbb{C}^n$. Hence it is compact. This implies that $\phi(S)$ is a compact set in $E$. Since 0 is not in $S$, 0 is not in $\phi(S)$. Since $E$ is hausdorff, $\phi(S)$ is closed and $E - \phi(S)$ is an open neighborhood of 0 in $E$. By continuity of multiplication by scalars at $(0, 0)$, there exists $\epsilon > 0$ and an open neighborhood $U$ of 0 in $E$ such that $zU \subset E - \phi(S)$, i.e., $zU \cap \phi(S) = \emptyset$ for all $|z| \leq \epsilon$.

Let $v \in U - \{0\}$. Then we have
\[ v = \sum_{i=1}^{n} c_i v_i. \]

Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$. Then, $\frac{1}{\|c\|} c \in S$, and $\frac{1}{\|c\|} v \in \phi(S)$. By our construction, we must have $\frac{1}{\|c\|} > \epsilon$. Hence, we have $\|c\| < \frac{1}{\epsilon}$, i.e., $c \in B_{\frac{1}{\epsilon}}$. This in turn yields $v \in \phi(\frac{1}{\epsilon} B_1)$. Therefore, we have
\[ \phi\left(\frac{1}{\epsilon} B_1\right) \supset U, \]
i.e., $\phi(B_1) \supset \epsilon U$. Hence, $\phi(B_1)$ is a neighborhood of 0 in $E$. This completes the proof of 2.1.2.

2.1.4. Corollary. Let $E$ be a hausdorff topological vector space over $\mathbb{C}$. Let $F$ be a finite-dimensional vector subspace of $E$. Then $F$ is closed in $E$.

Proof. Clearly, the topology of $E$ induces a structure of hausdorff topological vector space on $F$. Let $v_1, v_2, \ldots, v_n$ be a basis of $F$. Assume that $F$ is not closed. Let $w$ be a vector in the closure of $F$ which is not in $F$. Then $w$ is linearly independent of $v_1, v_2, \ldots, v_n$. Let $F'$ be the direct sum of $F$ and $\mathbb{C}w$. Then $F'$ is a $(n + 1)$-dimensional hausdorff topological vector space. By 2.1.2, we know that
\[ (c_1, c_2, \ldots, c_n, c_{n+1}) \mapsto \sum_{i=1}^{n} c_i v_i + c_{n+1}w \]
is an isomorphism of the topological vector space $\mathbb{C}^{n+1}$ onto $F'$. This isomorphism maps $\mathbb{C}^n \times \{0\}$ onto $F$. Therefore, $F$ is closed in $F'$, and $w$ is not in the closure of $F$. Hence, we have a contradiction. □

2.1.5. Lemma. Let $E$ and $F$ be two finite-dimensional hausdorff topological vector spaces. Then any linear map $\phi : E \to F$ is continuous.

2.2. Representations on topological vector spaces. Let $G$ be a compact group. Let $E$ be a hausdorff topological vector space over $\mathbb{C}$. We denote by $\text{GL}(E)$ the group of all automorphisms of $E$.

A (continuous) representation of $G$ on $E$ is a group homomorphism $\pi : G \to \text{GL}(E)$ such that $(g, v) \mapsto \pi(g)v$ is continuous from $G \times E$ into $E$. 
2.2.1. Lemma. Let $E$ be a Banach space and $\pi : G \to \text{GL}(E)$ a homomorphism such that $g \mapsto \pi(g)v$ is continuous function from $G$ into $E$ for all $v \in E$. Then $(\pi, E)$ is a representation of $G$ on $E$.

Proof. Assume that the function $g \mapsto \pi(g)v$ is continuous for any $v \in V$. Then the function $g \mapsto \|\pi(g)v\|$ is continuous on $G$. Since $G$ is compact, there exists $M$ such that $\|\pi(g)v\| < M$ for all $g \in G$. By Banach-Steinhaus theorem, we see that the function $g \mapsto \|\pi(g)\|$ is bounded on $G$.

Pick $C > 0$ such that $\|\pi(g)\| \leq C$ for all $g \in G$. Then we have

$$\|\pi(g)v - \pi(g')v'\| = \|\pi(g)v - \pi(g)v' + \pi(g')(v - v')\|$$

$$\leq \|\pi(g)v - \pi(g)v'\| + \|\pi(g')(v - v')\| \leq \|\pi(g)v - \pi(g)v'\| + C\|v - v'\|$$

for all $g, g' \in G$ and $v, v' \in E$. This clearly implies the continuity of the function $(g, v) \mapsto \pi(g)v$.

If $E$ is a finite-dimensional hausdorff topological vector space, by 2.1.5, any linear automorphism of $E$ is automatically an automorphism of topological vector spaces. Therefore $\text{GL}(E)$ is just the group of all linear automorphisms of $E$ as before.

Moreover, since the topology of $E$ is described by the euclidean norm and $E$ is a Banach space with respect to it, by 2.2.1, the only additional condition for a representation of $G$ is the continuity of the function $g \mapsto \pi(g)v$ for any $v \in E$. This implies the following result.

2.2.2. Lemma. Let $E$ be a finite-dimensional hausdorff topological vector space and $\pi$ a homomorphism of $G$ into $\text{GL}(E)$. Let $v_1, v_2, \ldots, v_n$ be a basis of $E$.

(i) $(\pi, E)$ is a representation of $G$ on $E$;

(ii) all matrix coefficients of $\pi(g)$ with respect to the basis $v_1, v_2, \ldots, v_n$ are continuous functions on $G$.

2.3. Algebra of matrix coefficients. Let $G$ be a compact group. The Banach space $C(G)$ is a commutative algebra with pointwise multiplication of functions, i.e., $(\psi, \phi) \mapsto \psi \cdot \phi$ where $(\psi \cdot \phi)(g) = \psi(g)\phi(g)$ for any $g \in G$.

First, we remark the following fact.

2.3.1. Lemma. $R$ and $L$ are representations of $G$ on $C(G)$.

Proof. Clearly, we have

$$\|R(g)\phi\| = \max_{h \in G} |(R(g)\phi)(h)| = \max_{h \in G} |\phi(hg)| = \max_{h \in G} |\phi(h)| = \|\phi\|.$$ 

Hence $R(g)$ is a continuous linear map on $C(G)$. Its inverse is $R(g^{-1})$, so $R(g) \in \text{GL}(C(G))$.

By 2.2.1, it is enough to show that the function $g \mapsto R(g)\phi$ is continuous for any function $\phi \in C(G)$.

By 1.1.1, $\phi$ is uniformly continuous, i.e., there exists a neighborhood $U$ of 1 in $G$ such that $g^{-1}g' \in U$ implies $|\phi(hg) - \phi(hg')| < \epsilon$ for all $h \in G$. Hence, we have

$$\|R(g)\phi - R(g')\phi\| = \max_{h \in G} |(R(g)\phi)(h) - (R(g')\phi)(h)| = \max_{h \in G} |\phi(hg) - \phi(hg')| < \epsilon$$

for $g' \in gU$. Hence, the function $g \mapsto R(g)\phi$ is continuous.

The proof for $L$ is analogous.

We say that the function $\phi \in C(G)$ is right (resp. left) $G$-finite if the vectors $\{R(g)\phi; g \in G\}$ (resp. $\{L(g)\phi; g \in G\}$) span a finite-dimensional subspace of $C(G)$.  


2.3.2. Lemma. Let $\phi \in C(G)$. The following conditions are equivalent.

(i) $\phi$ is left $G$-finite;
(ii) $\phi$ is right $G$-finite;
(iii) there exist $n$ and functions $a_i, b_i \in C(G), \ 1 \leq i \leq n$, such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Let $\phi$ be a right $G$-finite function. Then $\phi$ is in a finite-dimensional subspace $F$ invariant for $R$. The restriction of the representation $R$ to $F$ is continuous. Let $a_1, a_2, \ldots, a_n$ be a basis of $F$. Then, by 2.2.2, there exist $b_1, b_2, \ldots, b_n \in C(G)$ such that $R(g)\phi = \sum_{i=1}^{n} b_i(g)a_i$. Therefore we have

$$\phi(hg) = \sum_{i=1}^{n} b_i(g)a_i(h) = \sum_{i=1}^{n} a_i(h)b_i(g)$$

for all $h, g \in G$. Therefore (iii) holds.

If (iii) holds,

$$R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i$$

and $\phi$ is right $G$-finite.

Since the condition (iii) is symmetric, the equivalence of (i) and (iii) follows by applying the above argument to the opposite group of $G$. \hfill \Box

Therefore, we can call $\phi$ just a $G$-finite function in $C(G)$. Let $R(G)$ be the subset of all $G$-finite functions in $C(G)$.

2.3.3. Proposition. The set $R(G)$ is a subalgebra of $C(G)$.

Proof. Clearly, a multiple of a $G$-finite function is a $G$-finite function.

Let $\phi$ and $\psi$ be two $G$-finite functions. Then, by 2.3.2, there exists functions $a_i, b_i, c_i, d_i \in C(G)$ such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) \text{ and } \psi(gh) = \sum_{i=1}^{m} c_i(g)d_i(h)$$

for all $g, h \in G$. This implies that

$$(\phi + \psi)(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) + \sum_{i=1}^{m} c_i(g)d_i(h)$$

and

$$(\phi \cdot \psi)(gh) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(g)c_j(g)b_i(h)d_j(h) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot c_j)(g)(b_i \cdot d_i)(h)$$

for all $g, h \in G$. Hence, $\phi + \psi$ and $\phi \cdot \psi$ are $G$-finite. \hfill \Box

Clearly, $R(G)$ is an invariant subspace for $R$ and $L$.

The main result of this section is the following observation. Let $V$ be a finite-dimensional complex linear space and $\pi$ a continuous homomorphism of $G$ into $\text{GL}(V)$, i.e., $(\pi, V)$ is a representation of $G$. For $v \in V$ and $v^* \in V^*$ we call the continuous function $g \mapsto c_{v,v^*}(g) = \langle \pi(g)v, v^* \rangle$ a matrix coefficient of $(\pi, V)$.
2.3.4. Theorem. Let $\phi \in C(G)$. Then the following statements are equivalent:

(i) $\phi$ is in $R(G)$;

(ii) $\phi$ is a matrix coefficient of a finite-dimensional representation of $G$.

Proof. Let $(\pi, V)$ be a finite-dimensional representation of $G$. Let $v \in V$ and $v^* \in V^*$. By scaling $v^*$ if necessary, we can assume that $v$ is a vector in a basis of $V$ and $v^*$ a vector in the dual basis of $V^*$. Then, $c_{v, v^*}(g)$ is a matrix coefficient of the matrix of $\pi(g)$ in the basis of $V$. The rule of matrix multiplication implies that (iii) from 2.3.2 holds for $c_{v, v^*}$. Hence $\phi$ is $G$-finite.

Assume that $\phi$ is $G$-finite. Then, by 2.3.2, we have $R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i$ where $a_i, b_i \in C(G)$. We can also assume that $b_i$ are linearly independent. Let $V$ be the subspace of $R(G)$ spanned by $b_1, b_2, \ldots, b_n$. Then $V$ is a $G$-invariant subspace. Let $v = \phi$ and $v^* \in V^*$ such that $b_i(1) = \langle b_i, v^* \rangle$. Then

$$\langle R(g)v, v^* \rangle = \sum_{i=1}^{n} a_i(g)\langle b_i, v^* \rangle = \sum_{i=1}^{n} a_i(g)b_i(1) = \phi(g),$$

i.e., $\phi$ is a matrix coefficient of the restriction of $R$ to $V$. \hfill \Box

Therefore, we call $R(G)$ the algebra of matrix coefficients of $G$.

We also have the following stronger version of 2.3.2

2.3.5. Corollary. Let $\phi \in R(G)$. Then there exist $n$ and functions $a_i, b_i \in R(G)$, $1 \leq i \leq n$, such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Since $\phi$ is a matrix coefficient of a finite-dimensional representation by 2.3.4, the statement follows from the formula for the product of two matrices. \hfill \Box

Moreover, $R(G)$ has the following properties. For a function $\phi \in C(G)$ we denote by $\hat{\phi}$ the function $g \mapsto \overline{\phi(g)}$ on $G$; and by $\hat{\phi}$ the function $g \mapsto \phi(g^{-1})$.

2.3.6. Lemma. Let $\phi \in R(G)$. Then

(i) the function $\hat{\phi}$ is in $R(G)$;

(ii) the function $\hat{\phi}$ is in $R(G)$.

Proof. Obvious by 2.3.2. \hfill \Box

3. Some facts from functional analysis

3.1. Compact operators. Let $E$ be a Hilbert space and $T : E \to E$ a bounded linear operator.

We say that $T$ is a compact operator if $T$ is a bounded linear operator which maps the unit ball in $E$ into a relatively compact set.

3.1.1. Lemma. Compact operators for a two-sided ideal in the algebra of all bounded linear operators on $E$.

Proof. Let $S$ and $T$ be compact operators. Let $B$ be the unit ball in $E$. Then the images of $B$ in $E$ under $T$ and $S$ have compact closure. Hence, the image of $B \times B$ under $S \times T : E \times E \to E \times E$ has compact closure. Since the addition is a
continuous map from $E \times E$ into $E$, the image of $B$ under $S + T$ also has compact closure. Therefore, $S + T$ is a compact operator.

If $S$ is a bounded linear operator and $T$ a compact operator, the image of $B$ under $T$ has compact closure. Since $S$ is continuous, the image of $B$ under $ST$ also has compact closure. Hence, $ST$ is compact.

Analogously, the image of $B$ under $S$ is a bounded set since $S$ is bounded. Therefore, the image of $B$ under $TS$ has compact closure and $TS$ is also compact.

\[ \Box \]

3.2. Compact selfadjoint operators. Let $E$ be a Hilbert space. Let $T : E \to E$ be a nonzero compact selfadjoint operator.

3.2.1. Theorem. Either $\|T\|$ or $-\|T\|$ is an eigenvalue of $T$.

First we recall a simple fact.

3.2.2. Lemma. Let $u$ and $v$ be two nonzero vectors in $E$ such that $|(u|v)| = \|u\|\|v\|$. Then $u$ and $v$ are colinear.

Proof. Let $\lambda v$ be the orthogonal projection of $u$ to $v$. Then $u = \lambda v + w$ and $w$ is perpendicular to $v$. This implies that $\|u\|^2 = |\lambda|^2\|v\|^2 + \|w\|^2$. On the other hand, we have $\|u\| \cdot \|v\| = |(u|v)| = |\lambda|\|v\|^2$, i.e., $|\lambda| = \frac{\|u\|}{\|v\|}$. Hence, it follows that

$$\|u\|^2 = |\lambda|^2\|v\|^2 + \|w\|^2 = \|u\|^2 + \|w\|^2,$$

i.e., $\|w\|^2 = 0$ and $w = 0$. \[ \Box \]

Now we can prove the theorem. By rescaling $T$, we can assume that $\|T\| = 1$. Let $B$ be the unit ball in $E$. By our assumption, we know that

$$1 = \|T\| = \sup_{v \in B} \|Tv\|.$$

Therefore, there exists a sequence of vectors $v_n \in B$ such that $\lim_{n \to \infty} \|Tv_n\| = 1$. Since $T$ is compact, by going to a subsequence, we can also assume that $\lim_{n \to \infty} Tv_n = u$. This implies that

$$1 = \lim_{n \to \infty} \|Tv_n\| = \|u\|.$$

Moreover, we have $\lim_{n \to \infty} T^2v_n = Tu$. Hence, we have

$$1 = \|T\| : \|u\| \geq \|Tu\| = \lim_{n \to \infty} \|T^2v_n\| \geq \limsup_{n \to \infty} (\|T^2v_n\| : \|v_n\|) \geq \limsup_{n \to \infty} (T^2v_n : v_n) = \lim_{n \to \infty} (Tv_n : Tv_n) = \lim_{n \to \infty} \|Tv_n\|^2 = 1.$$

It follows that

$$\|Tu\| = 1.$$

Moreover, we have

$$1 = \|Tu\|^2 = (Tu|Tu) = (T^2u|u) \leq \|T^2u\|\|u\| \leq \|T^2\|\|u\|^2 \leq \|T\|^2\|u\|^2 = 1.$$

This finally implies that

$$(T^2u|u) = \|T^2u\|\|u\|.$$ 

By 3.2.2, it follows that $T^2u$ is proportional to $u$, i.e., $T^2u = \lambda u$. Moreover, we have

$$\lambda = \lambda(u|u) = (T^2u|u) = \|Tu\|^2 = 1.$$

It follows that $T^2u = u$. 

\[ \Box \]
Therefore, the linear subspace $F$ of $E$ spanned by $u$ and $Tu$ is $T$-invariant. Either $Tu = u$ or $v = \frac{1}{2}(u - Tu) \neq 0$. In the second case, we have $Tv = -v$.

This completes the proof of the existence of eigenvalues.

We need another fact.

3.2.3. Lemma. Let $T$ be a compact selfadjoint operator. Let $\lambda$ be an eigenvalue different from $0$. Then the eigenspace of $\lambda$ is finite-dimensional.

Proof. Assume that the corresponding eigenspace $V$ is infinite-dimensional. Then there would exist an orthonormal sequence $(e_n, n \in \mathbb{N})$ in $F$. Clearly, then the sequence $(Te_n, n \in \mathbb{N})$ would consist of mutually orthogonal vectors of length $|\lambda|$, hence it could not have compact closure in $V$, contradicting the compactness of $T$. Therefore, $V$ cannot be infinite-dimensional. □

3.3. An example. Denote by $\mu$ the Haar measure on $G$. Let $L^2(G)$ be the Hilbert space of square-integrable complex valued functions on $G$ with respect to the Haar measure $\mu$. We denote its norm by $\| \cdot \|_2$. Clearly, we have

$$\|f\|^2_2 = \int_G |f(g)|^2 \, d\mu(g) \leq \|f\|^2$$

for any $f \in C(G)$. Hence the inclusion $C(G) \rightarrow L^2(G)$ is a continuous map.

3.3.1. Lemma. The continuous linear map $i : C(G) \rightarrow L^2(G)$ is injective.

Proof. Let $f \in C(G)$ be such that $i(f) = 0$. This implies that $\|f\|_2 = 0$. On the other hand, the function $g \mapsto |f(g)|$ is a nonnegative continuous function on $G$. Assume that $M$ is the maximum of this function on $G$. If we would have $M > 0$, there would exist a nonempty open set $U \subset G$ such that $|f(g)| \geq \frac{M}{2}$ for $g \in U$. Therefore, we would have

$$\|f\|^2_2 = \int_G |f(g)|^2 \, d\mu(g) \geq \frac{M^2}{4} \mu(U) > 0,$$

by 1.3.16. Therefore, we must have $M = 0$. □

Since the measure of $G$ is 1, by Cauchy-Schwartz inequality, we have

$$\int_G |\phi(g)| \, d\mu(g) = \int_G 1 \cdot |\phi(g)| \, d\mu(g) \leq \|1\|_2 \cdot \|\phi\|_2 = \|\phi\|_2$$

for any $\phi \in L^2(\mu)$. Hence, $L^2(\mu) \subset L^1(\mu)$, where $L^1(\mu)$ is the Banach space of integrable functions on $G$.

Let $f$ be a continuous function on $G$. For any $\phi \in L^2(G)$, we put

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh) \, d\mu(h)$$

for $g \in G$.

By 1.1.1, $f$ is uniformly continuous on $G$. This implies that for any $\epsilon > 0$ there exists a symmetric neighborhood $U$ of 1 in $G$ such that $g'g^{-1} \in U$ implies $|f(g) - f(g')| < \epsilon$. Therefore, for arbitrary $h \in G$, we see that for $(g^{-1}h)(g^{-1}h)^{-1} = g^{-1}g \in U$ and we have

$$|f(g^{-1}h) - f(g^{-1}h)| < \epsilon.$$
This in turn implies that
\[
| (R(f)\phi)(g) - (R(f)\phi)(g') | = \left| \int_G f(h)\phi(gh)d\mu(h) - \int_G f(h)\phi(g'h)d\mu(h) \right|
= \int_G (f(g^{-1}h) - f(g'^{-1}h))\phi(h)d\mu(h) = \int_G |f(g^{-1}h) - f(g'^{-1}h)||\phi(h)|d\mu(h)
< \epsilon \cdot \int_G |\phi(h)|d\mu(h) \leq \epsilon \cdot \|\phi\|_2
\]
for any \( g' \in Ug \) and \( \phi \in L^2(G) \). This proves that functions \( R(f)\phi \) are in \( C(G) \) for any \( \phi \in L^2(G) \).

Moreover, by the invariance of Haar measure, we have
\[
| (R(f)\phi)(g) | \leq \int_G |f(h)||\phi(gh)|d\mu(h) \leq \|f\| \int_G |\phi(gh)|d\mu(h)
\leq \|f\| \int_G |\phi(h)|d\mu(h) \leq \|f\| \cdot \|\phi\|_2,
\]
it follows that
\[
\|R(f)\phi\| \leq \|f\| \cdot \|\phi\|_2
\]
for any \( \phi \in L^2(G) \). Hence, \( R(f) \) is a bounded linear operator from \( L^2(G) \) into \( C(G) \).

Hence the set \( S = \{ R(f)\phi \mid \|\phi\|_2 \leq 1 \} \) is bounded in \( C(G) \).

Clearly, the composition of \( R(f) \) with the natural inclusion \( i : C(G) \rightarrow L^2(G) \) is a continuous linear map from \( L^2(G) \) into itself which will denote by the same symbol. Therefore, the following diagram of continuous maps

\[
\begin{array}{ccc}
L^2(G) & \xrightarrow{R(f)} & L^2(G) \\
\downarrow{R(f)} & & \downarrow{R(f)} \\
C(G) & \xrightarrow{i} & L^2(G)
\end{array}
\]

is commutative.

We already remarked that \( S \) is a bounded set in \( C(G) \). Hence, \( S \) is a pointwise bounded family of continuous functions. In addition, by the above formula
\[
| (R(f)\phi)(g) - (R(f)\phi)(g') | < \epsilon,
\]
for all \( g' \in Ug \) and \( \phi \) in the unit ball in \( L^2(G) \). Hence, the set \( S \) is equicontinuous.

Hence we proved the following result.

3.3.2. Lemma. The set \( S \subset C(G) \) is pointwise bounded and equicontinuous.

By 1.2.1, the closure of the set \( S \) in \( C(G) \) is compact. Since \( i : C(G) \rightarrow L^2(G) \) is continuous, \( S \) has compact closure in \( L^2(G) \). Therefore, we have the following result.

3.3.3. Lemma. The linear operator \( R(f) : L^2(G) \rightarrow L^2(G) \) is compact.

Put \( f^*(g) = f(g^{-1}) \), \( g \in G \). Then \( f^* \in C(G) \).

3.3.4. Lemma. For any \( f \in C(G) \) we have
\[
R(f)^* = R(f^*).
\]
Proof. For $\phi, \psi \in L^2(G)$, we have, by 1.3.17,
\[
(R(f)\phi \mid \psi) = \int_G (R(f^*)\phi)(g)\overline{\psi(g)} \, d\mu(g) = \int_G \left( \int_G f(h)\phi(gh) \, d\mu(h) \right) \overline{\psi(g)} \, d\mu(g)
\]
\[
= \int_G f(h) \left( \int_G \phi(gh)\overline{\psi(g)} \, d\mu(g) \right) \, d\mu(h) = \int_G f(h) \left( \int_G \phi(g)\overline{\psi(gh^{-1})} \, d\mu(g) \right) \, d\mu(h)
\]
\[
= \int_G \phi(g) \left( \int_G f(h)\overline{\psi(gh^{-1})} \, d\mu(h) \right) \, d\mu(g) = \int_G \phi(g) \left( \int_G f^*(h^{-1})\psi(gh^{-1}) \, d\mu(h) \right) \, d\mu(g)
\]
\[
= \int_G \phi(g) \left( \int_G f^*(h)\psi(gh) \, d\mu(h) \right) \, d\mu(g) = (\phi \mid R(f^*)\psi).
\]

3.3.5. Corollary. The operator $R(f^*)R(f) = R(f)^*R(f)$ is a positive compact selfadjoint operator on $L^2(G)$.

4. Peter-Weyl theorem

4.1. $L^2$ version. Let $\phi \in L^2(G)$. Let $g \in G$. We put $(R(g)\phi)(h) = \phi(hg)$ for any $h \in G$. Then we have

\[
\|R(g)\phi\|^2 = \int_G |(R(g)\phi)(h)|^2 \, d\mu(h) = \int_G |\phi(hg)|^2 \, d\mu(h) = \int_G |\phi(h)|^2 \, d\mu(h) = \|\phi\|^2.
\]

Therefore, $R(g)$ is a continuous linear operator on $L^2(G)$. Clearly it is in $\text{GL}(L^2(G))$. Moreover, $R(g)$ is unitary.

Clearly, for any $g \in G$, the following diagram

\[
\begin{array}{ccc}
C(G) & \xrightarrow{R(g)} & C(G) \\
\downarrow i & & \downarrow i \\
L^2(G) & \xrightarrow{R(g)} & L^2(G)
\end{array}
\]

is commutative.

Analogously, we define $(L(g)\phi)(h) = \phi(g^{-1}h)$ for $h \in G$. Then $L(g)$ is a unitary operator on $L^2(G)$ which extends from $C(G)$.

Clearly, $R(g)$ and $L(h)$ commute for any $g, h \in G$.

4.1.1. Lemma. $L$ and $R$ are unitary representations of $G$ on $L^2(G)$.

Proof. It is enough to discuss $R$. The proof for $L$ is analogous.

Let $g \in G$ and $\phi \in L^2(G)$. We have to show that $h \mapsto R(h)\phi$ is continuous at $g$. Let $\epsilon > 0$. Since $C(G)$ is dense in $L^2(G)$, there exists $\psi \in C(G)$ such that $\|\phi - \psi\|_2 < \frac{\epsilon}{3}$. Since $R$ is a representation on $C(G)$, there exists a neighborhood $U$ of $g$ such that $h \in U$ implies $\|R(h)\psi - R(g)\psi\| < \frac{\epsilon}{3}$. This in turn implies that $\|R(h)\psi - R(g)\psi\|_2 < \frac{\epsilon}{3}$. Therefore we have

\[
\|R(h)\phi - R(g)\phi\|_2 \leq \|R(h)(\phi - \psi)\|_2 + \|R(h)\psi - R(g)\psi\|_2 + \|R(g)(\psi - \phi)\|_2 \\
\leq 2\|\phi - \psi\|_2 + \|R(h)\psi - R(g)\psi\|_2 < \epsilon
\]

for any $h \in U$. 

$\square$
Let $f$ be a continuous function on $G$. By 3.3.3, $R(f)$ is a compact operator on $L^2(G)$.

Let $\phi \in L^2(G)$. Then

$$
(R(f)L(g)\phi)(h) = \int_G f(k)(L(g)\phi)(hk) \, d\mu(k)
$$

and

$$
= \int_G f(k)(g^{-1}hk) \, d\mu(k) = (R(f)\phi)(g^{-1}h) = (L(g)R(f)\phi)(h)
$$

for all $g, h \in G$. Therefore, $R(f)$ commutes with $L(g)$ for any $g \in G$.

Let $F$ be the eigenspace of $R(f^*)R(f)$ for eigenvalue $\lambda > 0$. Then $F$ is finite-dimensional by 3.2.3.

4.1.2. **Lemma.** (i) Let $\phi \in F$. Then $\phi$ is a continuous function.

(ii) The vector subspace $F$ of $C(G)$ is in $R(G)$.

Proof. (i) The function $\phi$ is in the image of $R(f^*)$. Hence it is a continuous function.

(ii) By (i), $F \subseteq C(G)$. As we remarked above, the operator $R(f^*)R(f)$ commutes with the representation $L$. Therefore, the eigenspace $F$ is invariant subspace for $L$. Let $\phi$ be a function in $F$. Since $F$ is invariant for $L$, $\phi$ is $G$-finite. Hence, $\phi \in R(G)$. \hfill \Box

4.1.3. **Lemma.** The subspace $R(G)$ is invariant for $R(f)$.

Proof. Let $\phi \in R(G)$. By 2.3.5 we have

$$
(R(f)\phi)(g) = \int_G f(h)\phi(gh) \, d\mu(h) = \sum_{i=1}^{n} a_i(g) \int_G f(h)b_i(h) \, d\mu(h)
$$

for any $g \in G$, i.e., $R(f)\phi$ is a linear combination of $a_i$, $1 \leq i \leq n$. \hfill \Box

Let $E = R(G)^\perp$ in $L^2(G)$. Then, by 4.1.3, $R(G)$ is invariant for selfadjoint operator $R(f^*)R(f)$. This in turn implies that $E$ is also invariant for this operator. Therefore the restriction of this operator to $E$ is a positive selfadjoint operator. Assume that its norm is greater than 0. Then, by 3.2.1, the norm is an eigenvalue of this operator, and there exists a nonzero eigenvector $\phi \in E$ for that eigenvalue. Clearly, $\phi$ is an eigenvector for $R(f^*)R(f)$ too. By 4.1.2, $\phi$ is also in $R(G)$. Hence, we have $\|\phi\|^2_2 = (\phi | \phi) = 0$, and $\phi = 0$ in $L^2(G)$. Hence, we have a contradiction.

Therefore, the operator $R(f^*)R(f)$ is 0 when restricted to $E$. Hence

$$
0 = (R(f^*)R(f)\psi | \psi) = \|R(f)\psi\|^2
$$

for any $\psi \in E$. It follows that $R(f)\psi = 0$. Since $R(f)\psi$ is a continuous function, we have

$$
0 = (R(f)\psi)(1) = \int_G f(g)\psi(g) \, d\mu(g),
$$

i.e., $\psi$ is orthogonal to $f$.

Since $f \in C(G)$ was arbitrary and $C(G)$ is dense in $L^2(G)$, it follows that $\psi = 0$. This implies that $E = \{0\}$.

This completes the proof of the following result.

4.1.4. **Theorem** (Peter-Weyl). The algebra $R(G)$ is dense in $L^2(G)$. 

\[\Box\]
4.2. Continuous version. Let \( g \in G \). Assume that \( g \neq 1 \). Then there exists an open neighborhood \( U \) of 1 such that \( U \) and \( Ug \) are disjoint. There exists positive function \( \phi \) in \( C(G) \) such that \( \phi|_U = 0 \) and \( \phi|_{Ug} = 1 \). This implies that

\[
\|R(g)\phi - \phi\|^2 = \int_G |\phi(hg) - \phi(h)|^2 d\mu(h) \\
= \int_U |\phi(hg) - \phi(h)|^2 d\mu(h) + \int_{G-U} |\phi(hg) - \phi(h)|^2 d\mu(h) \geq \mu(U).
\]

Therefore \( R(g) \neq I \). Since by 4.1.4, \( R(G) \) is dense in \( L^2(G) \), \( R(g)|_{R(G)} \) is not the identity operator.

This implies the following result.

4.2.1. Lemma. Let \( g, g' \in G \) and \( g \neq g' \). Then there exists a function \( \phi \in R(G) \) such that \( \phi(g) \neq \phi(g') \).

Proof. Let \( h = g^{-1}g' \neq 1 \). Then there exists \( \psi \in R(G) \) such that \( R(h)\psi \neq \psi \). Hence, we have \( R(g)\psi \neq R(g')\psi \). It follows that \( \psi(hg) \neq \psi(hg') \) for some \( h \in G \). Therefore, the function \( \phi = L(h^{-1})\psi \) has the required property. \( \square \)

In other words, \( R(G) \) separates points in \( G \). By Stone-Weierstrass theorem, we have the following result which is a continuous version of Peter-Weyl theorem.

4.2.2. Theorem (Peter-Weyl). The algebra \( R(G) \) is dense in \( C(G) \).

Another consequence of 4.2.1 is the following result.

4.2.3. Lemma. Let \( U \) be an open neighborhood of 1 in \( G \). Then there exists a finite-dimensional representation \((\pi, V)\) of \( G \) such that \( \ker \pi \subset U \).

Proof. The complement \( G - U \) of \( U \) is a compact set. Since \( R(G) \) separates the points of \( G \), for any \( g \in G - U \) there exists a function \( \phi_g \in R(G) \) and an open neighborhood \( U_g \) of \( g \) such that \( |\phi_g(h) - \phi_g(1)| > \epsilon \) for \( h \in U_g \). Since \( G - U \) is compact, there exists a finite set \( g_1, g_2, \ldots, g_m \) in \( G - U \) such that \( U_{g_1}, U_{g_2}, \ldots, U_{g_m} \) form an open cover of \( G - U \) and \( |\phi_{g_i}(h) - \phi_{g_i}(1)| > \epsilon \) for \( h \in U_{g_i} \). Let \( \pi_i \) be a finite-dimensional representation of \( G \) with matrix coefficient \( \phi_{g_i} \). Then \( \pi_i(h) \neq I \) for \( h \in U_{g_i}, 1 \leq i \leq n \). Let \( \pi \) be the direct sum of \( \pi_i \). Then \( \pi(h) \neq I \) for \( h \in G - U \), i.e., \( \ker \pi \subset U \). \( \square \)

4.3. Matrix groups. Let \( G \) be a topological group. We say that \( G \) has no small subgroups if there exists a neighborhood \( U \) of 1 in \( G \) such that any subgroup of \( G \) contained in \( U \) is trivial.

4.3.1. Lemma. Let \( V \) be a finite-dimensional complex vector space. Then the group \( \text{GL}(V) \) has no small subgroups.

Proof. Let \( \mathcal{L}(V) \) be the space of all linear endomorphisms of \( V \). Then \( \exp : \mathcal{L}(V) \rightarrow \text{GL}(V) \) given by

\[
\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n
\]

defines a holomorphic map. Its differential at 0 is the identity map \( I \) on \( \mathcal{L}(V) \). Hence, by the inverse function theorem, it is a local diffeomorphism.

Let \( U \) be an open neighborhood of 1 in \( \text{GL}(V) \) and \( V \) the open ball around 0 in \( \mathcal{L}(V) \) of radius \( \epsilon \) (with respect to the linear operator norm) such that \( \exp : V \rightarrow U \)
is a diffeomorphism. Let $V'$ be the open ball of radius $\frac{\epsilon}{2}$ around 0 in $L(V)$. Then $U' = \exp(V')$ is an open neighborhood of 1 in $GL(V)$. Let $H$ be a subgroup of $GL(V)$ contained in $U'$. Let $S \in H$. Then $S = \exp(T)$ for some $T \in V'$. Hence, we have $S^2 = \exp(T)^2 = \exp(2T) \in H$. Moreover, $S^2 \in H$ and $S^2 = \exp(T')$ for some $T' \in V'$. It follows that $\exp(T') = \exp(2T')$ for $2T, T' \in V$. Since $\exp$ is injective on $V$, we must have $2T = T'$. Hence, $T \in \frac{1}{2}V'$. It follows that $H \subset \exp\left(\frac{1}{2}V'\right)$. By induction we get that $H \subset \exp\left(\frac{1}{2^n}V'\right)$ for any $n \in \mathbb{N}$. This implies that $H = \{1\}$. 

A compact subgroup of $GL(V)$ we call a compact matrix group.

4.3.2. Theorem. Let $G$ be a compact group. Then the following conditions are equivalent:

(i) $G$ has no small subgroups;

(ii) $G$ is isomorphic to a compact matrix group.

Proof. (i)$\Rightarrow$(ii) Let $U$ be an open neighborhood of $1 \in G$ such that it contains no nontrivial subgroups of $G$. By 4.2.3, there exists a finite-dimensional representation $(\pi, V)$ of $G$ such that $\ker \pi \subset U$. This clearly implies that $\ker \pi = \{1\}$, and $\pi : G \rightarrow GL(V)$ is an injective homomorphism. Since $G$ is compact, $\pi$ is homeomorphism of $G$ onto $\pi(G)$. Therefore, $G$ is isomorphic to the compact subgroup $\pi(G)$ of $GL(V)$.

(ii)$\Rightarrow$(i) Assume that $G$ is a compact subgroup of $GL(V)$. By 4.3.1, there exists an open neighborhood $U$ of $1$ in $GL(V)$ such that it contains no nontrivial subgroups. This implies that $G \cap U$ contains no nontrivial subgroups of $G$. □

4.3.3. Remark. For a compact matrix group $G$, since matrix coefficients of the natural representation separate points in $G$, 4.2.1 obviously holds. Therefore, in this situation, Stone-Weierstrass theorem immediately implies the second version of Peter-Weyl theorem, which in turn implies the first one.

4.3.4. Remark. By Cartan’s theorem [?], any compact matrix group is a Lie group. On the other hand, by [?] any Lie group has no small subgroups. Hence, compact Lie groups have no small subgroups and therefore they are compact matrix groups.

4.3.5. Remark. Let $T = \mathbb{R}/\mathbb{Z}$. Then $T$ is a compact abelian group. Let $G$ be the product of infinite number of copies of $T$. Then $G$ is a compact abelian group. By the definition of product topology, any neighborhood of 1 contains a nontrivial subgroup.

Let $G$ be an arbitrary compact group. Let $(\pi, V)$ be a finite-dimensional representation. Put $N = \ker \pi$. Then $N$ is a compact normal subgroup of $G$ and $G/N$ equipped with the quotient topology is a compact group. Clearly, $G/N$ is a compact matrix group.

Let $S$ be the family of all compact normal subgroups $N$ of $G$ such that $G/N$ is a compact matrix group. Clearly, $N, N' \in S$ implies $N \cap N' \in S$. Therefore, $S$ ordered by inclusion is a directed set. One can show that $G$ is a projective limit of the system $G/N, N \in S$. Therefore, any compact group is a projective limit of compact matrix groups. By the above remark, this implies that any compact group is a projective limit of compact Lie groups.