NOTES ON REPRESENTATIONS OF COMPACT GROUPS

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1. Haar measure on compact groups

1.1. **Compact groups.** Let G be a group. We say that G is a *topological group* if G is equipped with hausdorff topology such that the multiplication $(g, h) \longmapsto gh$ from the product space $G \times G$ into G and the inversion $g \longmapsto g^{-1}$ from G into G are continuous functions.

Let G and H be two topological groups. A *morphism of topological groups* $\varphi : G \longrightarrow H$ is a group homomorphism which is also continuous.

Topological groups and morphisms of topological groups for the category of topological groups.

Let G be a topological group. Let G^{opp} be the topological space G with the multiplication $(g, h) \longmapsto g \star h = h \cdot g$. Then G^{opp} is also a topological group which we call the *opposite group* of G. Clearly, the inverse of an element $g \in G$ is the same as the inverse in G^{opp}. Moreover, the map $g \longmapsto g^{-1}$ is an isomorphism of G with G^{opp}. Clearly, we have $(G^{opp})^{opp} = G$.

A topological group G is *compact*, if G is a compact space. The opposite group of a compact group is compact.

We shall need the following fact. Let G be a topological group. We say that a function $\phi : G \longrightarrow \mathbb{C}$ is *right (resp. left) uniformly continuous* on G if for any $\epsilon > 0$ there exists an open neighborhood U of 1 such that $|\phi(g) - \phi(h)| < \epsilon$ for any $g, h \in G$ such that $gh^{-1} \in U$ (resp. $g^{-1}h \in U$). Clearly, a right uniformly continuous function on G is left uniformly continuous function on G^{opp}.

1.1.1. **Lemma.** *Let G be a compact group. Let ϕ be a continuous function on G. Then ϕ is right and left uniformly continuous on G.*

Proof. By the above discussion, it is enough to prove that ϕ is right uniformly continuous.

Let $\epsilon > 0$. Let consider the set $A = \{(g, g') \in G \times G \mid |\phi(g) - \phi(g')| < \epsilon\}$. Then A is an open set in $G \times G$. Let U be an open neighborhood of 1 in G and $B_U = \{(g, g') \in G \times G \mid g'g^{-1} \in U\}$. Since the function $(g, g') \longmapsto g'g^{-1}$ is continuous on $G \times G$ the set B_U is open. It is enough to show that there exists an open neighborhood V of 1 in G such that $B_V \subset A$.

Clearly, B_U are open sets containing the diagonal Δ in $G \times G$. Moreover, under the homomorphism κ of $G \times G$ given by $\kappa(g, g') = (g, g'g^{-1})$, $g, g' \in G$, the sets B_U correspond to the sets $G \times U$. In addition, the diagonal Δ corresponds to $G \times \{1\}$. Assume that the open set O corresponds to A.

By the definition of product topology, for any $g \in G$ there exist neighborhoods U_g of 1 and V_g of g such that $V_g \times U_g$ is a neighborhood of $(g, 1)$ contained in O. Clearly, $(V_g; g \in G)$ is an open cover of G. Since G is compact, there exists a finite subcovering $(V_{g_i}; 1 \leq i \leq n)$ of G. Put $U = \bigcap_{i=1}^{n} U_{g_i}$. Then U is an open neighborhood of 1 in G. Moreover, $G \times U$ is an open set in $G \times G$ contained in O. Therefore $B_U \subset A$. $\square$

Therefore, we can say that a continuous function on G is uniformly continuous.

1.2. **A compactness criterion.** Let X be a compact space. Denote by $C(X)$ the space of all complex valued continuous functions on X. Let $\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. Then $f \longmapsto \|f\|$ is a norm on $C(X)$, $C(X)$ is a Banach space.

Let $\mathcal{S}$ be a subset of $C(X)$.

We say that $\mathcal{S}$ is *equicontinuous* if for any $\epsilon > 0$ and $x \in X$ there exists a neighborhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{S}$.

We say that $\mathcal{S}$ is *pointwise bounded* if for any $x \in X$ there exists $M > 0$ such that $|f(x)| \le M$ for all $f \in \mathcal{S}$.

The aim of this section is to establish the following theorem.

1.2.1. **Theorem** (Arzelà-Ascoli)**.** *Let $\mathcal{S}$ be a pointwise bounded, equicontinuous subset of $C(X)$. Then the closure of $\mathcal{S}$ is a compact subset of $C(X)$.*

Proof. We first prove that $\mathcal{S}$ is bounded in $C(X)$. Let $\epsilon > 0$. Since $\mathcal{S}$ is equicontinuous, for any $x \in X$, there exists an open neighborhood U_x of x such that $y \in U_x$ implies that $|f(y) - f(x)| < \epsilon$ for all $f \in \mathcal{S}$. Since X is compact, there exists a finite set of points $x_1, x_2, \dots, x_n \in X$ such that $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ cover X.

Since $\mathcal{S}$ is pointwise bounded, there exists $M \ge 2\epsilon$ such that $|f(x_i)| \le \frac{M}{2}$ for all $1 \le i \le n$ and all $f \in \mathcal{S}$. Let $x \in X$. Then $x \in U_{x_i}$ for some $1 \le i \le n$. Therefore, we have

$$|f(x)| \le |f(x) - f(x_i)| + |f(x_i)| < \frac{M}{2} + \epsilon \le M$$

for all $f \in S$. It follows that $\|f\| \le M$ for all $f \in \mathcal{S}$. Hence $\mathcal{S}$ is contained in a closed ball of radius M centered at 0 in $C(X)$.

Now we prove that $\mathcal{S}$ is contained in a finite family of balls of fixed small radius centered in elements of $\mathcal{S}$. We keep the choices from the first part of the proof. Let $D = \{z \in \mathbb{C} \mid |z| \le M\}$. Then D is compact. Consider the compact set D^n. It has natural metric given by $d(z, y) = \max_{1 \le i \le n} |z_i - y_i|$. There exist points $\alpha_1, \alpha_2, \dots, \alpha_m$ in D^n such that the balls $B_i = \{\beta \in D^n \mid d(\alpha_i, \beta) < \epsilon\}$ cover D^n.

Denote by Φ the map from $\mathcal{S}$ into D^n given by $f \longmapsto (f(x_1), f(x_2), \dots, f(x_n))$. Then we can find a subfamily of the above cover of D^n consisting of balls intersecting $\Phi(\mathcal{S})$. After a relabeling, we can assume that these balls are B_i for $1 \le i \le k$. Let $f_1, f_2, \dots, f_k$ be functions in $\mathcal{S}$ such that $\Phi(f_i)$ is in the ball B_i for any $1 \le i \le k$. Denote by C_i the open ball of radius 2ϵ centered in $\Phi(f_i)$. Let $\beta \in B_i$. Then we have $d(\beta, \alpha_i) < \epsilon$ and $d(\Phi(f_i), \alpha_i) < \epsilon$. Hence, we have $d(\beta, \Phi(f_i)) < 2\epsilon$, i.e., $B_i \subset C_i$. It follows that $\Phi(\mathcal{S})$ is contained in the union of $C_1, C_2, \dots, C_k$.

Differently put, for any function $f \in \mathcal{S}$, there exists $1 \le i \le k$ such that $|f(x_j) - f_i(x_j)| < 2\epsilon$ for all $1 \le j \le n$.

Let $x \in X$. Then $x \in U_{x_j}$ for some $1 \le j \le n$. Hence, we have

$$|f(x) - f_i(x)| \le |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| < 4\epsilon,$$

i.e., $\|f - f_i\| < 4\epsilon$.

Now we can prove the compactness of the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$. Assume that $\bar{\mathcal{S}}$ is not compact. Then there exists an open cover $\mathcal{U}$ of $\bar{\mathcal{S}}$ which doesn't contain a finite subcover. By the above remark, $\bar{\mathcal{S}}$ can be covered by finitely many closed balls $\{f \in C(X) \mid \|f - f_i\| \le 1\}$ with $f_i \in \mathcal{S}$. Therefore, there exists a set K_1

which is the intersection of $\bar{\mathcal{S}}$ with one of the closed balls and which is not covered by a finite subcover of $\mathcal{U}$. By induction, we can construct a decreasing family $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ of closed subsets of $\bar{\mathcal{S}}$ which are contained in closed balls of radius $\frac{1}{n}$ centered in some point of $\mathcal{S}$, such that none of K_n is covered by a finite subcover of $\mathcal{U}$.

Let $(F_n; n \in \mathbb{N})$ be a sequence of functions such that $F_n \in K_n$ for all $n \in \mathbb{N}$. Then $F_p, F_q \in K_n$ for all p, q greater than n. Since K_n are contained in closed balls of radius $\frac{1}{n}$, $\|F_p - F_q\| \leq \frac{2}{n}$ for all p, q greater than n. Hence, (F_n) is a Cauchy sequence in $C(X)$. Therefore, it converges to a function $F \in C(X)$. This function is in $\bar{\mathcal{S}}$ and therefore in one element V of the open cover $\mathcal{U}$. Therefore, for sufficiently large n, there exists a closed ball of radius $\frac{2}{n}$ centered in F which is contained in V. Since F is also in K_n, we see that K_n is in V. This clearly contradicts our construction of K_n. It follows that $\bar{\mathcal{S}}$ must be compact. □

1.3. **Haar measure on compact groups.** Let $\mathcal{C}_{\mathbb{R}}(G)$ be the space of real valued functions on G. For any function $f \in \mathcal{C}_{\mathbb{R}}(G)$ we define the *maximum* $M(f) = \max_{g \in G} f(g)$ and *minimum* $m(f) = \min_{g \in G} f(g)$. Moreover, we denote by $V(f) = M(f) - m(f)$ the *variation* of f.

Clearly, the function f is constant on G if and only if $V(f) = 0$.

Let $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$ be two functions such that $\|f - f'\| < \epsilon$. Then

$$f(g) - \epsilon < f'(g) < f(g) + \epsilon$$

for all $g \in G$. This implies that

$$m(f) - \epsilon < f'(g) < M(f) + \epsilon$$

for all $g \in G$, and

$$m(f) - \epsilon < m(f') < M(f') < M(f) + \epsilon.$$

Hence

$$V(f') = M(f') - m(f') < M(f) - m(f) + 2\epsilon = V(f) + 2\epsilon,$$

i.e., $V(f') - V(f) < 2\epsilon$. By symmetry, we also have $V(f) - V(f') < 2\epsilon$. It follows that $|V(f) - V(f')| < 2\epsilon$.

Therefore, we have the following result.

1.3.1. **Lemma.** *The variation V is a continuous function on $\mathcal{C}_{\mathbb{R}}(G)$.*

Let $f \in \mathcal{C}_{\mathbb{R}}(G)$ and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ a finite sequence of points in G. We define the *(right) mean value* $\mu(f, \mathbf{a})$ of f with respect to $\mathbf{a}$ as

$$\mu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(g a_i)$$

for all $g \in G$. Clearly, $\mu(f, \mathbf{a})$ is a continuous real function on G.

If f is a constant function, $\mu(f, \mathbf{a}) = f$.

Clearly, mean value $f \longmapsto \mu(f, \mathbf{a})$ is a linear map. Moreover, we have the following result.

1.3.2. **Lemma.** (i) *The linear map $f \longmapsto \mu(f, \mathbf{a})$ is continuous. More precisely, we have*

$$\|\mu(f, \mathbf{a})\| \leq \|f\|$$

for any $f \in C_{\mathbb{R}}(G)$;

(ii)
$$M(\mu(f,\mathbf{a})) \leq M(f)$$
for any $f \in C_{\mathbb{R}}(G)$*;*

(iii)
$$m(\mu(f,\mathbf{a})) \geq m(f)$$
for any $f \in C_{\mathbb{R}}(G)$*;*

(iv)
$$V(\mu(f,\mathbf{a})) \leq V(f)$$
for any $f \in C_{\mathbb{R}}(G)$*.*

Proof. (i) Clearly, we have
$$\|\mu(f,\mathbf{a})\| = \max_{g\in G} |\mu(f,\mathbf{a})| \leq \frac{1}{n}\sum_{g\in G} \max_{g\in G} |f(ga_i)| = \|f\|.$$

(ii) We have
$$M(\mu(f,\mathbf{a})) = \frac{1}{n}\max_{g\in G}\left(\sum_{i=1}^{n} f(ga_i)\right) \leq \frac{1}{n}\sum_{i=1}^{n}\max_{g\in G} f(ga_i) = M(f).$$

(iii) We have
$$m(\mu(f,\mathbf{a})) = \frac{1}{n}\min_{g\in G}\left(\sum_{i=1}^{n} f(ga_i)\right) \geq \frac{1}{n}\sum_{i=1}^{n}\min_{g\in G} f(ga_i) = m(f).$$

(iv) By (ii) and (iii), we have
$$V(\mu(f,\mathbf{a})) = M(\mu(f,\mathbf{a})) - m(\mu(f,\mathbf{a})) \leq M(f) - m(f) = V(f).$$
□

Denote by $\mathcal{M}_f$ the set of mean values of f for all finite sequences in G.

1.3.3. **Lemma.** *The set of functions* $\mathcal{M}_f$ *is uniformly bounded and equicontinuous.*

Proof. By 1.3.2 (ii) and (iii), it follows that
$$m(f) \leq m(\mu(f,\mathbf{a})) \leq \mu(f,\mathbf{a})(g) \leq M(\mu(f,\mathbf{a})) \leq M(f).$$
This implies that $\mathcal{M}_f$ is uniformly bounded.

Now we want to prove that $\mathcal{M}_f$ is equicontinuous. First, by 1.1.1, the function f is uniformly continuous. Hence, for any $\epsilon > 0$, there exists an open neighborhood U of 1 in G such that $|f(g) - f(h)| < \epsilon$ if $gh^{-1} \in U$. Since, this implies that $(ga_i)(ha_i)^{-1} = gh^{-1} \in U$ for any $1 \leq i \leq n$, we see that
$$|\mu(f,\mathbf{a})(g) - \mu(f,\mathbf{a})(h)| = \frac{1}{n}\left|\sum_{i=1}^{n}(f(ga_i) - f(ha_i))\right| \leq \frac{1}{n}\sum_{i=1}^{n}|f(ga_i) - f(ha_i)| < \epsilon$$
for $g \in hU$. Hence, the family $\mathcal{M}_f$ is equicontinuous. □

By 1.2.1, we have the following consequence.

1.3.4. **Lemma.** *The set* $\mathcal{M}_f$ *of all right mean values of* f *has compact closure in* $\mathcal{C}_{\mathbb{R}}(G)$*.*

We need another result on variation of mean value functions. Clearly, if f is a constant function $\mu(f,\mathbf{a}) = f$ for any $\mathbf{a}$.

1.3.5. **Lemma.** *Let f be a function in $\mathcal{C}_{\mathbb{R}}(G)$. Assume that f is not a constant. Then there exists* $\mathbf{a}$ *such that $V(\mu(f,\mathbf{a})) < V(f)$.*

Proof. Since f is not constant, we have $m(f) < M(f)$. Let C be such that $m(f) < C < M(f)$. Then there exists an open set V in G such that $f(g) \leq C$ for all $g \in V$. Since the right translates of V cover G, by compactness of G we can find $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ such that $(Va_i^{-1}, 1 \leq i \leq n)$ is an open cover of G. For any $g \in Va_i^{-1}$ we have $ga_i \in V$ and $f(ga_i) \leq C$. Hence, we have

$$\mu(f,\mathbf{a})(g) = \frac{1}{n}\sum_{j=1}^{n} f(ga_j) = \frac{1}{n}\left(f(ga_i) + \sum_{j\neq i} f(ga_j)\right) \leq \frac{1}{n}(C + (n-1)M(f)) < M(f).$$

On the other hand, by 1.3.2.(iii) we know that $m(\mu(f,\mathbf{a})) \geq m(f)$ for any $\mathbf{a}$. Hence we have

$$V(\mu(f,\mathbf{a})) = M(\mu(f,\mathbf{a})) - m(\mu(f,\mathbf{a})) < M(f) - m(f) = V(f).$$

□

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_m)$ be two finite sequences in G. We define $\mathbf{a}\cdot\mathbf{b} = (a_i b_j; 1 \leq i \leq n, 1 \leq j \leq m)$.

1.3.6. **Lemma.** *We have*

$$\mu(\mu(f,\mathbf{b}),\mathbf{a}) = \mu(f,\mathbf{b}\cdot\mathbf{a}).$$

Proof. We have

$$\mu(\mu(f,\mathbf{b}),\mathbf{a}) = \frac{1}{m}\sum_{i=1}^{m} \mu(f,\mathbf{b})(ga_i) = \frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n} f(ga_i b_j) = \mu(f,\mathbf{a}\cdot\mathbf{b}).$$

□

1.3.7. **Lemma.** *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{M}_f$ contains a constant function on G.*

Proof. By 1.3.4, we know that $\overline{\mathcal{M}_f}$ is compact. Since, by 1.3.1, the variation V is continuous on $\mathcal{C}_{\mathbb{R}}(G)$, it attains its minimum α at some $\varphi \in \overline{\mathcal{M}_f}$.

Assume that φ is not a constant. By 1.3.5, there exists $\mathbf{a}$ such that $V(\mu(\varphi,\mathbf{a})) < V(\varphi)$. Let $\alpha - V(\mu(\varphi,\mathbf{a})) = \epsilon > 0$.

Since V and $\mu(\cdot,\mathbf{a})$ are continuous maps by 1.3.1 and 1.3.2.(i), this implies that there is $\mathbf{b}$ such that $|V(\mu(\varphi,\mathbf{a})) - V(\mu(\mu(f,\mathbf{b}),\mathbf{a}))| < \frac{\epsilon}{2}$. Therefore, we have

$$V(\mu(\mu(f,\mathbf{b}),\mathbf{a})) \leq V(\mu(\varphi,\mathbf{a})) + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2}.$$

By 1.3.6, we have

$$V(\mu(f,\mathbf{a}\cdot\mathbf{b})) < \alpha - \frac{\epsilon}{2}$$

contrary to our choice of α.

It follows that φ is a constant function. In addition $\alpha = 0$. □

Consider now left mean values of a function $f \in \mathcal{C}_{\mathbb{R}}(G)$. We define the left mean value of f with respect to $\mathbf{a} = (a_1, a_2, \dots, a_n)$ as the function

$$\nu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(a_i g)$$

for $g \in G$. We denote my $\mathcal{N}_f$ the set of all left mean values of f.

Let G^{opp} be the compact group opposite to G. Then the left mean values of f on G are the right mean values of f on G^{opp}.

Hence, from 1.3.7, we deduce the following result.

1.3.8. **Lemma.** *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{N}_f$ contains a constant function on G.*

By direct calculation we get the following result.

1.3.9. **Lemma.** *For any $f \in C_{\mathbb{R}}(G)$ we have*

$$\nu(\mu(f, \mathbf{a}), \mathbf{b}) = \mu(\nu(f, \mathbf{b}), \mathbf{a})$$

for any two finite sequences $\mathbf{a}$ *and* $\mathbf{b}$ *in G.*

Proof. We have

$$\nu(\mu(f, \mathbf{a}), \mathbf{b})(g) = \frac{1}{m} \sum_{j=1}^{m} \mu(f, \mathbf{a})(b_j g) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(b_j g a_i)$$
$$= \frac{1}{n} \sum_{i=1}^{n} \nu(f, \mathbf{b})(g a_i) = \mu(\nu(f, \mathbf{b}), \mathbf{a})(g)$$

for any $g \in G$. □

Putting together these results, we finally get the following.

1.3.10. **Proposition.** *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{M}_f$ contains a unique function constant on G.*

This function is also the unique constant function in $\mathcal{N}_f$.

Proof. Let φ and ψ be two constant functions such that φ is in the closure of $\mathcal{M}_f$ and ψ is in the closure of $\mathcal{N}_f$. For any $\epsilon > 0$ we have $\mathbf{a}$ and $\mathbf{b}$ such that $\|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}$ and $\|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}$.

On the other hand, we have

$$\|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \nu(\varphi, \mathbf{b})\|$$
$$= \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| \leq \|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}.$$

In the same way. we also have

$$\|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| = \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \mu(\psi, \mathbf{a})\|$$
$$= \|\mu(\nu(f, \mathbf{b}) - \psi, \mathbf{a})\| \leq \|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

By 1.3.9, this immediately yields

$$\|\varphi - \psi\| \leq \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| + \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| < \epsilon.$$

This implies that $\varphi = \psi$. Therefore, any constant function in the closure of $\mathcal{M}_f$ has to be equal to ψ. □

The value of the unique constant function in the closure of $\mathcal{M}_f$ is denoted by $\mu(f)$ and called *the mean value* of f on G. In this way, we get a function $f \longmapsto \mu(f)$ on $\mathcal{C}_{\mathbb{R}}(G)$.

Let $\gamma : \mathcal{C}_{\mathbb{R}}(G) \longrightarrow \mathbb{R}$ be a linear form. We say that γ is *positive* if for any $f \in \mathcal{C}_{\mathbb{R}}(G)$ such that $f(g) \geq 0$ for any $g \in G$ we have $\gamma(f) \geq 0$.

1.3.11. **Lemma.** *The function μ is a positive linear form on $\mathcal{C}_{\mathbb{R}}(G)$.*

To prove this result we need some preparation.

1.3.12. **Lemma.** *Let $f \in C_{\mathbb{R}}(G)$. Then, for any $\mathbf{a}$ we have*

$$\mu(\mu(f, \mathbf{a})) = \mu(f).$$

Proof. Let $\mu(f) = \alpha$. Let φ be the function equal to α everywhere on G. Fix $\epsilon > 0$. Then there exists a finite sequence $\mathbf{b}$ such that

$$\|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This implies that

$$\|\nu(f - \varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \nu(\varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This, by 1.3.2.(i), implies that

$$\|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| \leq \|\nu(f - \varphi, \mathbf{b})\| < \epsilon$$

for any finite sequence $\mathbf{a}$.

By 1.3.9, we have

$$\|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| = \|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| < \epsilon,$$

and

$$\|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| = \|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| < \epsilon.$$

Therefore, if we fix $\mathbf{a}$, we see that φ is in the closure of $\mathcal{N}_{\mu(f,\mathbf{a})}$. By 1.3.10, this proves our assertion. □

Let f and f' be two functions in $C_{\mathbb{R}}(G)$. Let $\alpha = \mu(f)$ and $\beta = \mu(f')$. Denote by φ and ψ the corresponding constant functions. Let $\epsilon > 0$.

Clearly. there exists $\mathbf{a}$ such that

$$\|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}.$$

This, by 1.3.2.(ii) implies, that we have

$$\|\mu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\mu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| < \frac{\epsilon}{2}$$

for arbitrary $\mathbf{b}$. By 1.3.6, this in turn implies that

$$\|\mu(f, \mathbf{a} \cdot \mathbf{b}) - \varphi\| < \frac{\epsilon}{2}.$$

On the other hand, by 1.3.12, we have $\mu(\mu(f', \mathbf{a})) = \mu(f') = \beta$. Therefore, there exists a finite sequence $\mathbf{b}$ such that

$$\|\mu(\mu(f', \mathbf{a}), \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

This, by 1.3.6, implies that

$$\|\mu(f', \mathbf{a} \cdot \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

Hence, we have

$$\|\mu(f+f', \mathbf{a}\cdot\mathbf{b}) - (\varphi+\psi)\| \leq \|\mu(f, \mathbf{a}\cdot\mathbf{b}) - \varphi\| + \|\mu(f', \mathbf{a}\cdot\mathbf{b}) - \psi\| < \epsilon.$$

Therefore, $\varphi + \psi$ is in the closure of $\mathcal{M}_{f+f'}$. It follows that

$$\mu(f+f') = \alpha + \beta = \mu(f) + \mu(f'),$$

i.e., μ is additive.

Let $c \in \mathbb{R}$ and $f \in C_{\mathbb{R}}(G)$. Then $\mu(cf, \mathbf{a}) = c\mu(f, \mathbf{a})$ for any $\mathbf{a}$. Therefore, $\mathcal{M}_{cf} = c\mathcal{M}_f$. This immediately implies that $\mu(cf) = c\mu(f)$. Therefore μ is a linear form.

Assume that f is a function in $C_{\mathbb{R}}(G)$ such that $f(g) \geq 0$ for all $g \in G$. Then $\mu(f, \mathbf{a})(g) \geq 0$ for any $\mathbf{a}$ and $g \in G$. Hence, any function $\phi \in \mathcal{M}_f$ satisfies $\phi(g) \geq 0$ for all $g \in G$. This immediately implies that $\phi(g) \geq 0$, $g \in G$, for any ϕ in the closure of $\mathcal{M}_f$. It follows that $\mu(f) \geq 0$. Hence, we μ is a positive linear form. This completes the proof of 1.3.11.

Clearly, $\mu(1) = 1$. Let $f \in \mathcal{C}_{\mathbb{R}}(G)$. Then we have

$$-\|f\| \leq f(g) \leq \|f\|$$

for any $g \in G$. Since μ is a positive linear form, we see that

$$-\|f\| = \mu(-\|f\|) \leq \mu(f) \leq \mu(\|f\|) = \|f\|.$$

Therefore, we have

$$|\mu(f)| \leq \|f\|$$

for any $f \in \mathcal{C}_{\mathbb{R}}(G)$. In particular, μ is a continuous linear form on $\mathcal{C}_{\mathbb{R}}(G)$.

By Riesz representation theorem, the linear form $\mu : C_{\mathbb{R}}(G) \longrightarrow \mathbb{R}$ defines a regular positive measure μ on G such that

$$\mu(f) = \int_G f \, d\mu.$$

Clearly, we have

$$\mu(G) = \int_G d\mu = \mu(1) = 1.$$

so we say that μ is *normalized*.

Denote by R (resp. L) the *right regular representation* (resp. *left regular representation* of G on $C(G)$ given by $(R(g)f)(h) = f(hg)$ (resp. $(L(g)f)(h) = f(g^{-1}h)$) for any $f \in C(G)$ and $g, h \in G$.

1.3.13. **Lemma.** *Let $f \in C_{\mathbb{R}}(G)$ and $g \in G$. Then*

$$\mu(R(g)f) = \mu(L(g)f) = \mu(f).$$

Proof. Let $\mathbf{g} = (g)$. Clearly, we have

$$\mu(f, \mathbf{g})(h) = f(hg) = (R(g)f)(h)$$

for all $h \in G$, i.e., $R(g)f = \mu(f, \mathbf{g})$. By 1.3.12, we have

$$\mu(R(g)f) = \mu(\mu(f, \mathbf{g})) = \mu(f).$$

This statement for G^{opp} implies the other equality. $\square$

We say that the linear form μ is *biinvariant*, i.e., *right invariant* and *left invariant*.

The above result implies that the measure μ is *biinvariant*, i.e., we have the following result.

1.3.14. **Lemma.** *Let A be a measurable set in G. Then gA and Ag are also measurable and*

$$\mu(gA) = \mu(Ag) = \mu(A)$$

for any $g \in G$.

Proof. Since $C_{\mathbb{R}}(G)$ is dense in $L^1(\mu)$, the invariance from 1.3.13 holds for any function $f \in L^1(\mu)$. Applying it to the characteristic function of the set A implies the result. □

A normalized biinvariant positive measure μ on G is called a *Haar measure* on G.

We proved the existence part of the following result.

1.3.15. **Theorem.** *Let G be a compact group. Then there exists a unique Haar measure μ on G.*

Proof. We constructed a Haar measure on G.

It remains to prove the uniqueness. Let ν be another Haar measure on G. Then, by left invariance, we have

$$\int_G \mu(f, \mathbf{a})\, d\nu = \frac{1}{n} \sum_{i=1}^{n} \int_G f(ga_i)\, d\nu(g) = \int_G f\, d\nu$$

for any $\mathbf{a}$. Hence the integral with respect to ν is constant on $\mathcal{M}_f$. By continuity, it is also constant on its closure. Therefore, we have

$$\int_G f\, d\nu = \mu(f) \int_G d\nu = \mu(f) = \int_G f\, d\mu$$

for any $C_{\mathbb{R}}(G)$. This in turn implies that $\nu = \mu$. □

1.3.16. **Lemma.** *Let μ be the Haar measure on G. Let U be a nonempty open set in G. Then $\mu(U) > 0$.*

Proof. Since U is nonempty, $(Ug; g \in G)$ is an open cover of G. It contains a finite subcover $(Ug_i; 1 \le i \le n)$. Therefore we have

$$1 = \mu(G) = \mu\left(\bigcup_{i=1}^{n} Ug_i\right) \le \sum_{i=1}^{n} \mu(Ug_i) = \sum_{i=1}^{n} \mu(U) = n\,\mu(U)$$

by 1.3.14. This implies that $\mu(U) \ge \frac{1}{n}$. □

1.3.17. **Lemma.** *Let f be a continuous function on G. Then*

$$\int_G f(g^{-1})\, d\mu(g) = \int_G f(g)\, d\mu(g).$$

Proof. Clearly, it is enough to prove the statement for real-valued functions. Therefore, we can consider the linear form $\nu : f \longmapsto \int_G f(g^{-1})\, d\mu(g)$. Clearly, this a positive continuous linear form on $C_{\mathbb{R}}(G)$. Moreover,

$$\begin{aligned}\nu(f) = \int_G f(h^{-1})\, d\mu(h) &= \int_G f((hg)^{-1})\, d\mu(h) \\ &= \int_G f(g^{-1}h^{-1})\, d\mu(h) = \int_G (L(g)f)(h^{-1})\, d\mu(h) = \nu(L(g)f)\end{aligned}$$

and

$$\nu(f) = \int_G f(h^{-1})\,d\mu(h) = \int_G f((g^{-1}h)^{-1})\,d\mu(h)$$
$$= \int_G f(h^{-1}g)\,d\mu(h) = \int_G (R(g)f)(h^{-1})\,d\mu(h) = \nu(R(g)f)$$

for any $g \in G$. Hence, this linear form is left and right invariant. By the uniqueness of the Haar measure we get the statement. □

2. Algebra of matrix coefficients

2.1. **Topological vector spaces.** Let E be a vector space over $\mathbb{C}$. We say that E is a *topological vector space* over $\mathbb{C}$, if it is also equipped with a topology such that the functions $(u, v) \longmapsto u + v$ from $E \times E$ into E, and $(\alpha, u) \longmapsto \alpha u$ from $\mathbb{C} \times E$ into E are continuous.

A morphism $\varphi : E \longrightarrow F$ of topological vector space E into F is a continuous linear map from E to F.

We say that E is a hausdorff topological vector space if the topology of E is hausdorff.

The vector space $\mathbb{C}^n$ with its natural topology is a hausdorff topological vector space.

Let E be a topological vector space and F a vector subspace of E. Then F is a topological vector space with the induced topology. Moreover, if E is hausdorff, F is also hausdorff.

2.1.1. **Lemma.** *Let E be a topological vector space over $\mathbb{C}$. Then the following conditions are equivalent:*

(i) *E is hausdorff;*
(ii) *$\{0\}$ is a closed set in E.*

Proof. Assume that E is hausdorff. Let $v \in E$, $v \neq 0$. Then there exist open neighborhoods U of 0 and V of v such that $U \cap V = \emptyset$. In particular, $V \subset E - \{0\}$. Hence, $E - \{0\}$ is an open set. This implies that $\{0\}$ is closed.

Assume now that $\{0\}$ is closed in E. Then $E - \{0\}$ is an open set. Let u and v be different vectors in E. Then $u - v \neq 0$. Since the function $(x, y) \longmapsto x + y$ is continuous, there exist open neighborhoods U of u and V of v such that $U - V \subset E - \{0\}$. This in turn implies that $U \cap V = \emptyset$. □

The main result of this section is the following theorem. It states that hausdorff finite-dimensional topological vector spaces have unique topology.

2.1.2. **Theorem.** *Let E be a finite-dimensonal hausdorff topological vector space over $\mathbb{C}$. Let $v_1, v_2, \dots, v_n$ be a basis of E. Then the linear map $\mathbb{C}^n \longrightarrow E$ given by*

$$(c_1, c_2, \dots, c_n) \longmapsto \sum_{i=1}^n c_i v_i$$

is an isomorphism of topological vector spaces.

Clearly, the map

$$\phi(z) = \sum_{i=1}^n z_i v_i,$$

for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, is a continuous linear isomorphism of $\mathbb{C}^n$ onto E. Therefore, it is enough to show that that map is also open.

We start with an elementary lemma.

2.1.3. **Lemma.** *Let E and F be topological vector spaces and $\phi : E \longrightarrow F$ a linear map. Let $(U_i, i \in I)$ be a fundamental system of neighborhoods of 0 in E. If $\phi(U_i)$, $i \in I$, are neighborhoods of 0 in F, ϕ is an open map.*

Proof. Let U be an open set in E. For any $u \in U$ there exists $i \in I$ such that $u + U_i$ is a neighborhood of u contained in U. Therefore, $\phi(u + U_i) = \phi(u) + \phi(U_i)$ is a neighborhood of $\phi(u)$ contained in $\phi(U)$. Hence, $\phi(u)$ is an interior point of $\phi(U)$. This implies that $\phi(U)$ is open. □

We consider on $\mathbb{C}^n$ the standard euclidean norm $\|\cdot\|$. Let $B_1 = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the open unit ball in $\mathbb{C}^n$. Then the balls $\{\epsilon B_1 \mid 0 < \epsilon < \infty\}$ form a fundamental system of neighborhoods of 0 in $\mathbb{C}^n$. By 2.1.3, to show that the above map $\phi : \mathbb{C}^n \longrightarrow E$ is open it is enough to show that $\phi(B_1)$ is a neighborhood of 0 in E.

Let $S = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Then, S is a bounded and closed set in $\mathbb{C}^n$. Hence it is compact. This implies that $\phi(S)$ is a compact set in E. Since 0 is not in S, 0 is not in $\phi(S)$. Since E is hausdorff, $\phi(S)$ is closed and $E - \phi(S)$ is an open neighborhood of 0 in E. By continuity of multiplication by scalars at $(0, 0)$, there exists $\epsilon > 0$ and an open neighborhood U of 0 in E such that $zU \subset E - \phi(S)$, i.e., $zU \cap \phi(S) = \emptyset$ for all $|z| \leq \epsilon$.

Let $v \in U - \{0\}$. Then we have

$$v = \sum_{i=1}^{n} c_i v_i.$$

Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$. Then, $\frac{1}{\|c\|} c \in S$, and $\frac{1}{\|c\|} v \in \phi(S)$. By our construction, we must have $\frac{1}{\|c\|} > \epsilon$. Hence, we have $\|c\| < \frac{1}{\epsilon}$, i.e., $c \in B_{\frac{1}{\epsilon}}$. This in turn yields $v \in \phi(\frac{1}{\epsilon} B_1)$. Therefore, we have

$$\phi\left(\frac{1}{\epsilon} B_1\right) \supset U,$$

i.e., $\phi(B_1) \supset \epsilon U$. Hence, $\phi(B_1)$ is a neighborhood of 0 in E. This completes the proof of 2.1.2.

2.1.4. **Corollary.** *Let E be a hausdorff topological vector space over $\mathbb{C}$. Let F be a finite-dimensional vector subspace of E. Then F is closed in E.*

Proof. Clearly, the topology of E induces a structure of hausdorff topological vector space on F. Let $v_1, v_2, \dots, v_n$ be a basis of F. Assume that F is not closed. Let w be a vector in the closure of F which is not in F. Then w is linearly independent of $v_1, v_2, \dots, v_n$. Let F' be the direct sum of F and $\mathbb{C}w$. Then F' is a $(n+1)$-dimensional hausdorff topological vector space. By 2.1.2, we know that

$$(c_1, c_2, \dots, c_n, c_{n+1}) \longmapsto \sum_{i=1}^{n} c_i v_i + c_{n+1} w$$

is an isomorphism of the topological vector space $\mathbb{C}^{n+1}$ onto F'. This isomorphism maps $\mathbb{C}^n \times \{0\}$ onto F. Therefore, F is closed in F', and w is not in the closure of F. Hence, we have a contradiction. □

2.1.5. **Lemma.** *Let E and F be two finite-dimensional hausdorff topological vector spaces. Then any linear map $\phi : E \longrightarrow F$ is continuous.*

2.2. **Representations on topological vector spaces.** Let G be a compact group. Let E be a hausdorff topological vector space over $\mathbb{C}$. We denote by $\mathrm{GL}(E)$ the group of all automorphisms of E.

A (continuous) representation of G on E is a group homomorphism $\pi : G \longrightarrow \mathrm{GL}(E)$ such that $(g, v) \longmapsto \pi(g)v$ is continuous from $G \times E$ into E.

2.2.1. **Lemma.** *Let E be a Banach space and $\pi : G \longrightarrow \mathrm{GL}(E)$ a homomorphism such that $g \longmapsto \pi(g)v$ is continuous function from G into E for all $v \in E$. Then (π, E) is a representation of G on E.*

Proof. Assume that the function $g \longmapsto \pi(g)v$ is continuous for any $v \in V$. Then the function $g \longmapsto \|\pi(g)v\|$ is continuous on G. Since G is compact, there exists M such that $\|\pi(g)v\| < M$ for all $g \in G$. By Banach-Steinhaus theorem, we see that the function $g \longmapsto \|\pi(g)\|$ is bounded on G.

Pick $C > 0$ such that $\|\pi(g)\| \le C$ for all $g \in G$. Then we have

$$\begin{aligned}\|\pi(g)v - \pi(g')v'\| &= \|(\pi(g)v - \pi(g')v) + \pi(g')(v - v')\| \\ &\le \|\pi(g)v - \pi(g')v\| + \|\pi(g')\|\|v - v'\| \le \|\pi(g)v - \pi(g')v\| + C\|v - v'\|\end{aligned}$$

for all $g, g' \in G$ and $v, v' \in E$. This clearly implies the continuity of the function $(g, v) \longmapsto \pi(g)v$. □

If E is a finite-dimensional hausdorff topological vector space, by 2.1.5, any linear automorphism of E is automatically an automorphism of topological vector spaces. Therefore $\mathrm{GL}(E)$ is just the group of all linear automorphisms of E as before.

Moreover, since the topology of E is described by the euclidean norm and E is a Banach space with respect to it, by 2.2.1, the only additional condition for a representation of G is the continuity of the function $g \longmapsto \pi(g)v$ for any $v \in E$. This implies the following result.

2.2.2. **Lemma.** *Let E be a finite-dimensional hausdorff topological vector space and π a homomorphism of G into $\mathrm{GL}(E)$. Let $v_1, v_2, \dots, v_n$ be a basis of E.*

(i) *(π, E) is a representation of G on E;*
(ii) *all matrix coefficients of $\pi(g)$ with respect to the basis $v_1, v_2, \dots, v_n$ are continuous functions on G.*

2.3. **Algebra of matrix coefficients.** Let G be a compact group. The Banach space $C(G)$ is an commutative algebra with pointwise multiplication of functions, i.e., $(\psi, \phi) \longmapsto \psi \cdot \phi$ where $(\psi \cdot \phi)(g) = \psi(g)\phi(g)$ for any $g \in G$.

First, we remark the following fact.

2.3.1. **Lemma.** *R and L are representations of G on $C(G)$.*

Proof. Clearly, we have

$$\|R(g)\phi\| = \max_{h \in G} |(R(g)\phi)(h)| = \max_{h \in G} |\phi(hg)| = \max_{h \in G} |\phi(h)| = \|\phi\|.$$

Hence $R(g)$ is a continuous linear map on $C(G)$. Its inverse is $R(g^{-1})$, so $R(g) \in \mathrm{GL}(C(G))$.

By 2.2.1, it is enough to show that the function $g \longmapsto R(g)\phi$ is continuous for any function $\phi \in C(G)$.

By 1.1.1, ϕ is uniformly continuous, i.e., there exists a neighborhood U of 1 in G such that $g^{-1}g' \in U$ implies $|\phi(hg) - \phi(hg')| < \epsilon$ for all $h \in G$. Hence, we have

$$\|R(g)\phi - R(g')\phi\| = \max_{h\in G} |(R(g)\phi)(h) - (R(g')\phi)(h)| = \max_{h\in G} |\phi(hg) - \phi(hg')| < \epsilon$$

for $g' \in gU$. Hence, the function $g \longrightarrow R(g)\phi$ is continuous.

The proof for L is analogous. □

We say that the function $\phi \in C(G)$ is right (resp. left) *G-finite* if the vectors $\{R(g)\phi; g \in G\}$ (resp. $\{L(g)\phi; g \in G\}$) span a finite-dimensional subspace of $C(G)$.

2.3.2. **Lemma.** *Let $\phi \in C(G)$. The following conditions are equivalent.*

(i) *ϕ is left G-finite;*

(ii) *ϕ is right G-finite;*

(iii) *there exist n and functions $a_i, b_i \in C(G)$, $1 \le i \le n$, such that*

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Let ϕ be a right G-finite function. Then ϕ is in a finite-dimensional subspace F invariant for R. The restriction of the representation R to F is continuous. Let $a_1, a_2, \dots, a_n$ be a basis of F. Then, by 2.2.2, there exist $b_1, b_2, \dots, b_n \in C(G)$ such that $R(g)\phi = \sum_{i=1}^{n} b_i(g)a_i$. Therefore we have

$$\phi(hg) = \sum_{i=1}^{n} b_i(g)a_i(h) = \sum_{i=1}^{n} a_i(h)b_i(g)$$

for all $h, g \in G$. Therefore (iii) holds.

If (iii) holds,

$$R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i$$

and ϕ is right G-finite.

Since the condition (iii) is symmetric, the equivalence of (i) and (iii) follows by applying the above argument to the opposite group of G. □

Therefore, we can call ϕ just a *G-finite* function in $C(G)$. Let $R(G)$ be the subset of all G-finite functions in $C(G)$.

2.3.3. **Proposition.** *The set $R(G)$ is a subalgebra of $C(G)$.*

Proof. Clearly, a multiple of a G-finite function is a G-finite function.

Let ϕ and ψ be two G-finite functions. Then, by 2.3.2, there exists functions $a_i, b_i, c_i, d_i \in C(G)$ such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) \text{ and } \psi(gh) = \sum_{i=1}^{m} c_i(g)d_i(h)$$

for all $g, h \in G$. This implies that

$$(\phi + \psi)(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) + \sum_{i=1}^{m} c_i(g)d_i(h)$$

and

$$(\phi \cdot \psi)(gh) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(g) c_j(g) b_i(h) d_j(h) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot c_j)(g)(b_i \cdot d_i)(h)$$

for all $g, h \in G$. Hence, $\phi + \psi$ and $\phi \cdot \psi$ are G-finite. $\square$

Clearly, $R(G)$ is an invariant subspace for R and L.

The main result of this section is the following observation. Let V be a finite-dimensional complex linear space and π a continuous homomorphism of G into $\mathrm{GL}(V)$, i.e., (π, V) is a *representation* of G. For $v \in V$ and $v \in V^*$ we call the continuous function $g \longmapsto c_{v,v*}(g) = \langle \pi(g)v, v^* \rangle$ a *matrix coefficient* of (π, V).

2.3.4. **Theorem.** *Let $\phi \in C(G)$. Then the following statements are equivalent:*

(i) *ϕ is in $R(G)$;*
(ii) *ϕ is a matrix coefficient of a finite-dimensional representation of G.*

Proof. Let (π, V) be a finite-dimensional representation of G. Let $v \in V$ and $v^* \in V^*$. By scaling v^* if necessary, we can assume that v is a vector in a basis of V and v^* a vector in the dual basis of V^*. Then, $c_{v,v^*}(g)$ is a matrix coefficient of the matrix of $\pi(g)$ in the basis of V. The rule of matrix multiplication implies that (iii) from 2.3.2 holds for c_{v,v^*}. Hence ϕ is G-finite.

Assume that ϕ is G-finite. Then, by 2.3.2, we have $R(g)\phi = \sum_{i=1}^{n} a_i(g) b_i$ where $a_i, b_i \in C(G)$. We can also assume that b_i are linearly independent. Let V be the subspace of $R(G)$ spanned by $b_1, b_2, \dots, b_n$. Then V is a G-invariant subspace. Let $v = \phi$ and $v^* \in V^*$ such that $b_i(1) = \langle b_i, v^* \rangle$. Then

$$\langle R(g)v, v^* \rangle = \sum_{i=1}^{n} a_i(g) \langle b_i, v^* \rangle = \sum_{i=1}^{n} a_i(g) b_i(1) = \phi(g),$$

i.e., ϕ is a matrix coefficient of the restriction of R to V. $\square$

Therefore, we call $R(G)$ the *algebra of matrix coefficients* of G.

We also have the following stronger version of 2.3.2

2.3.5. **Corollary.** *Let $\phi \in R(G)$. Then there exist n and functions $a_i, b_i \in R(G)$, $1 \le i \le n$, such that*

$$\phi(gh) = \sum_{i=1}^{n} a_i(g) b_i(h)$$

for all $g, h \in G$.

Proof. Since ϕ is a matrix coefficient of a finite-dimensional representation by 2.3.4, the statement follows from the formula for the product of two matrices. $\square$

Moreover, $R(G)$ has the following properties. For a function $\phi \in C(G)$ we denote by $\bar{\phi}$ the function $g \longmapsto \overline{f(g)}$ on G; and by $\hat{\phi}$ the function $g \longmapsto f(g^{-1})$.

2.3.6. **Lemma.** *Let $\phi \in R(G)$. Then*

(i) *the function $\bar{\phi}$ is in $R(G)$;*
(ii) *the function $\hat{\phi}$ is in $R(G)$.*

Proof. Obvious by 2.3.2. $\square$

3. Some facts from functional analysis

3.1. **Compact operators.** Let E be a Hilbert space and $T : E \longrightarrow E$ a bounded linear operator.

We say that T is a *compact operator* if T is a bounded linear operator which maps the unit ball in E into a relatively compact set.

3.1.1. **Lemma.** *Compact operators for a two-sided ideal in the algebra of all bounded linear operators on E.*

Proof. Let S and T be compact operators. Let B be the unit ball in E. Then the images of B in E under T and S have compact closure. Hence, the image of $B \times B$ under $S \times T : E \times E \longrightarrow E \times E$ has compact closure. Since the addition is a continuous map from $E \times E$ into E, the image of B under $S + T$ also has compact closure. Therefore, $S + T$ is a compact operator.

If S is a bounded linear operator and T a compact operator, the image of B under T has compact closure. Since S is continuous, the image of B under ST also has compact closure. Hence, ST is compact.

Analogously, the image of B under S is a bounded set since S is bounded. Therefore, the image of B under TS has compact closure and TS is also compact. □

3.2. **Compact selfadjoint operators.** Let E be a Hilbert space. Let $T : E \longrightarrow E$ be a nonzero compact selfadjoint operator.

3.2.1. **Theorem.** *Either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

First we recall a simple fact.

3.2.2. **Lemma.** *Let u and v be two nonzero vectors in E such that $|(u|v)| = \|u\| \cdot \|v\|$. Then u and v are colinear.*

Proof. Let λv be the orthogonal projection of u to v. Then $u = \lambda v + w$ and w is perpendicular to v. This implies that $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2$. On the other hand, we have $\|u\| \cdot \|v\| = |(u|v)| = |\lambda| \|v\|^2$, i.e., $|\lambda| = \frac{\|u\|}{\|v\|}$. Hence, it follows that

$$\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2 = \|u\|^2 + \|w\|^2,$$

i.e., $\|w\|^2 = 0$ and $w = 0$. □

Now we can prove the theorem. By rescaling T, we can assume that $\|T\| = 1$.

Let B be the unit ball in E. By our assumption, we know that

$$1 = \|T\| = \sup_{v \in B} \|Tv\|.$$

Therefore, there exists a sequence of vectors $v_n \in B$ such that $\lim_{n\to\infty} \|Tv_n\| = 1$. Since T is compact, by going to a subsequence, we can also assume that $\lim_{n\to\infty} Tv_n = u$. This implies that

$$1 = \lim_{n\to\infty} \|Tv_n\| = \|u\|.$$

Moreover, we have $\lim_{n\to\infty} T^2 v_n = Tu$. Hence, we have

$$\begin{aligned} 1 = \|T\| \cdot \|u\| \geq \|Tu\| = \lim_{n\to\infty} \|T^2 v_n\| \geq \limsup_{n\to\infty} (\|T^2 v_n\| \cdot \|v_n\|) \\ \geq \limsup_{n\to\infty} (T^2 v_n | v_n) = \lim_{n\to\infty} (Tv_n | Tv_n) = \lim_{n\to\infty} \|Tv_n\|^2 = 1. \end{aligned}$$

It follows that
$$\|Tu\| = 1.$$
Moreover, we have
$$1 = \|Tu\|^2 = (Tu|Tu) = (T^2u|u) \le \|T^2u\|\|u\| \le \|T^2\|\|u\|^2 \le \|T\|^2\|u\|^2 = 1.$$
This finally implies that
$$(T^2u|u) = \|T^2u\|\|u\|.$$
By 3.2.2, it follows that T^2u is proportional to u, i.e. $T^2u = \lambda u$. Moreover, we have
$$\lambda = \lambda(u|u) = (T^2u|u) = \|Tu\|^2 = 1.$$
It follows that $T^2u = u$.

Therefore, the linear subspace F of E spanned by u and Tu is T-invariant. Either $Tu = u$ or $v = \frac{1}{2}(u - Tu) \neq 0$. In the second case, we have $Tv = -v$.

This completes the proof of the existence of eigenvalues.

We need another fact.

3.2.3. **Lemma.** *Let T be a compact selfadjoint operator. Let λ be an eigenvalue different from 0. Then the eigenspace of λ is finite-dimensional.*

Proof. Assume that the corresponding eigenspace V is infinite-dimensional. Then there would exist an orthonormal sequence $(e_n, n \in \mathbb{N})$ in F. Clearly, then the sequence $(Te_n, n \in \mathbb{N})$ would consist of mutually orthogonal vectors of length $|\lambda|$, hence it could not have compact closure in V, contradicting the compactness of T. Therefore, V cannot be infinite-dimensional. □

3.3. **An example.** Denote by μ the Haar measure on G. Let $L^2(G)$ be the Hilbert space of square-integrable complex valued functions on G with respect to the Haar measure μ. We denote its norm by $\|\cdot\|_2$. Clearly, we have
$$\|f\|_2^2 = \int_G |f(g)|^2 \, d\mu(g) \le \|f\|^2$$
for any $f \in C(G)$. Hence the inclusion $C(G) \longrightarrow L^2(G)$ is a continuous map.

3.3.1. **Lemma.** *The continuous linear map $i : C(G) \longrightarrow L^2(G)$ is injective.*

Proof. Let $f \in C(G)$ be such that $i(f) = 0$. This implies that $\|f\|_2 = 0$. On the other hand, the function $g \longmapsto |f(g)|$ is a nonnegative continuous function on G. Assume that M is the maximum of this function on G. If we would have $M > 0$, there would exist a nonempty open set $U \subset G$ such that $|f(g)| \ge \frac{M}{2}$ for $g \in U$. Therefore, we would have
$$\|f\|_2^2 = \int_G |f(g)|^2 \, d\mu(g) \ge \frac{M^2}{4}\mu(U) > 0,$$
by 1.3.16. Therefore, we must have $M = 0$. □

Since the measure of G is 1, by Cauchy-Schwartz inequality, we have
$$\int_G |\phi(g)| \, d\mu(g) = \int_G 1 \cdot |\phi(g)| \, d\mu(g) \le \|1\|_2 \cdot \|\phi\|_2 = \|\phi\|_2$$
for any $\phi \in L^2(\mu)$. Hence, $L_2(G) \subset L_1(G)$, where $L_1(G)$ is the Banach space of integrable functions on G.

Let f be a continuous function on G. For any $\phi \in L^2(G)$, we put

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)d\mu(h)$$

for $g \in G$.

By 1.1.1, f is uniformly continuous on G. This implies that for any $\epsilon > 0$ there exists a neighborhood U of 1 in G such that $g'g^{-1} \in U$ implies $|f(g) - f(g')| < \epsilon$. Therefore, for arbitrary $h \in G$, we see that for $(g'^{-1}h)(g^{-1}h)^{-1} = g'^{-1}g \in U$ and we have

$$|f(g^{-1}h) - f(g'^{-1}h)| < \epsilon.$$

This in turn implies that

$$\begin{aligned}
&|(R(f)\phi)(g) - (R(f)\phi)(g')| = \left| \int_G f(h)\phi(gh)d\mu(h) - \int_G f(h)\phi(g'h)\,d\mu(h) \right| \\
&= \left| \int_G (f(g^{-1}h) - f(g'^{-1}h))\phi(h)\,d\mu(h) \right| = \int_G |f(g^{-1}h) - f(g'^{-1}h)|\,|\phi(h)|\,d\mu(h) \\
&\qquad\qquad < \epsilon \cdot \int_G |\phi(h)|\,d\mu(h) \le \epsilon \cdot \|\phi\|_2
\end{aligned}$$

for any $g' \in Ug$ and ϕ in $L^2(G)$. This proves that functions $R(f)\phi$ are in $C(G)$ for any $\phi \in L^2(G)$.

Moreover, by the invariance of Haar measure, we have

$$\begin{aligned}
|(R(f)\phi)(g)| &\le \int_G |f(h)||\phi(gh)|\,d\mu(h) \le \|f\| \int_G |\phi(gh)|\,d\mu(h) \\
&\le \|f\| \int_G |\phi(h)|\,d\mu(h) \le \|f\| \cdot \|\phi\|_2,
\end{aligned}$$

it follows that

$$\|R(f)\phi\| \le \|f\| \cdot \|\phi\|_2$$

for any $\phi \in L^2(G)$. Hence, $R(f)$ is a bounded linear operator from $L^2(G)$ into $C(G)$.

Hence the set $\mathcal{S} = \{R(f)\phi \mid \|\phi\|_2 \le 1\}$ is bounded in $C(G)$.

Clearly, the composition of $R(f)$ with the natural inclusion $i : C(G) \longrightarrow L^2(G)$ is a continuous linear map from $L^2(G)$ into itself which will denote by the same symbol. Therefore, the following diagram of continuous maps

$$\begin{tikzcd}
L^2(G) \arrow[r, "R(f)"] \arrow[d, "R(f)"'] & L^2(G) \\
C(G) \arrow[ur, "i"'] &
\end{tikzcd}$$

is commutative.

We already remarked that $\mathcal{S}$ is a bounded set in $C(G)$. Hence, $\mathcal{S}$ is a pointwise bounded family of continuous functions. In addition, by the above formula

$$|(R(f)\phi)(g) - (R(f)\phi)(g')| < \epsilon,$$

for all $g' \in Ug$ and ϕ in the unit ball in $L^2(G)$. Hence, the set $\mathcal{S}$ is equicontinuous.

Hence we proved the following result.

3.3.2. **Lemma.** *The set $\mathcal{S} \subset C(G)$ is pointwise bounded and equicontinuous.*

By 1.2.1, the closure of the set $\mathcal{S}$ in $C(G)$ is compact. Since $i : C(G) \longrightarrow L^2(G)$ is continuous, $\mathcal{S}$ has compact closure in $L^2(G)$. Therefore, we have the following result.

3.3.3. **Lemma.** *The linear operator $R(f) : L^2(G) \longrightarrow L^2(G)$ is compact.*

Put $f^*(g) = \overline{f(g^{-1})}$, $g \in G$. Then $f^* \in C(G)$.

3.3.4. **Lemma.** *For any $f \in C(G)$ we have*

$$R(f)^* = R(f^*).$$

Proof. For $\phi, \psi \in L^2(G)$, we have, by 1.3.17,

$$\begin{aligned}
(R(f)\phi \mid \psi) &= \int_G (R(f)\phi)(g)\overline{\psi(g)}\, d\mu(g) = \int_G \left(\int_G f(h)\phi(gh)\, d\mu(h) \right) \overline{\psi(g)}\, d\mu(g) \\
&= \int_G f(h) \left(\int_G \phi(gh)\overline{\psi(g)}\, d\mu(g) \right) d\mu(h) = \int_G f(h) \left(\int_G \phi(g)\overline{\psi(gh^{-1})}\, d\mu(g) \right) d\mu(h) \\
&= \int_G \phi(g) \left(\overline{\int_G \overline{f(h)}\psi(gh^{-1})\, d\mu(h)} \right) d\mu(g) \\
&= \int_G \phi(g) \left(\overline{\int_G f^*(h^{-1})\psi(gh^{-1})\, d\mu(h)} \right) d\mu(g) \\
&= \int_G \phi(g) \left(\overline{\int_G f^*(h)\psi(gh)\, d\mu(h)} \right) d\mu(g) = (\phi \mid R(f^*)\psi).
\end{aligned}$$

□

3.3.5. **Corollary.** *The operator $R(f^*)R(f) = R(f)^*R(f)$ is a positive compact selfadjoint operator on $L^2(G)$.*

4. Peter-Weyl theorem

4.1. **L^2 version.** Let $\phi \in L^2(G)$. Let $g \in G$. We put $(R(g)\phi)(h) = \phi(hg)$ for any $h \in G$. Then we have

$$\|R(g)\phi\|_2^2 = \int_G |(R(g)\phi)(h)|^2\, d\mu(h) = \int_G |\phi(hg)|^2\, d\mu(h) = \int_G |\phi(h)|^2\, d\mu(h) = \|\phi\|_2^2.$$

Therefore, $R(g)$ is a continuous linear operator on $L^2(G)$. Clearly it is in $\mathrm{GL}(L^2(G))$. Moreover, $R(g)$ is unitary.

Clearly, for any $g \in G$, the following diagram

$$\begin{array}{ccc}
C(G) & \xrightarrow{R(g)} & C(G) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
L^2(G) & \xrightarrow[R(g)]{} & L^2(G)
\end{array}$$

is commutative.

Analogously, we define $(L(g)\phi)(h) = \phi(g^{-1}h)$ for $h \in G$. Then $L(g)$ is a unitary operator on $L^2(G)$ which extends from $C(G)$.

Clearly, $R(g)$ and $L(h)$ commute for any $g, h \in G$.

4.1.1. **Lemma.** *L and R are unitary representations of G on $L^2(G)$.*

Proof. It is enough to discuss R. The proof for L is analogous.

Let $g \in G$ and $\phi \in L^2(G)$. We have to show that $h \longmapsto R(h)\phi$ is continuous at g. Let $\epsilon > 0$. Since $C(G)$ is dense in $L^2(G)$, there exists $\psi \in C(G)$ such that $\|\phi - \psi\|_2 < \frac{\epsilon}{3}$. Since R is a representation on $C(G)$, there exists a neighborhood U of g such that $h \in U$ implies $\|R(h)\psi - R(g)\psi\| < \frac{\epsilon}{3}$. This in turn implies that $\|R(h)\psi - R(g)\psi\|_2 < \frac{\epsilon}{3}$. Therefore we have

$$\begin{aligned}\|R(h)\phi - R(g)\phi\|_2 &\le \|R(h)(\phi - \psi)\|_2 + \|R(h)\psi - R(g)\psi\|_2 + \|R(g)(\psi - \phi)\|_2 \\ &\le 2\|\phi - \psi\|_2 + \|R(h)\psi - R(g)\psi\|_2 < \epsilon\end{aligned}$$

for any $h \in U$. □

Let f be a continuous function on G. By 3.3.3, $R(f)$ is a compact operator on $L^2(G)$.

Let $\phi \in L^2(G)$. Then

$$\begin{aligned}(R(f)L(g)\phi)(h) &= \int_G f(k)(L(g)\phi)(hk)\,d\mu(k) \\ &= \int_G f(k)\phi(g^{-1}hk)\,d\mu(k) = (R(f)\phi)(g^{-1}h) = (L(g)R(f)\phi)(h)\end{aligned}$$

for all $g, h \in G$. Therefore, $R(f)$ commutes with $L(g)$ for any $g \in G$.

Let F be the eigenspace of $R(f^*)R(f)$ for eigenvalue $\lambda > 0$. Then F is finite-dimensional by 3.2.3.

4.1.2. **Lemma.** (i) *Let $\phi \in F$. Then ϕ is a continuous function.*
(ii) *The vector subspace F of $C(G)$ is in $R(G)$.*

Proof. (i) The function ϕ is in the image of $R(f^*)$. Hence it is a continuous function.

(ii) By (i), $F \subset C(G)$. As we remarked above, the operator $R(f^*)R(f)$ commutes with the representation L. Therefore, the eigenspace F is invariant subspace for L. Let ϕ be a function in F. Since F is invariant for L, ϕ is G-finite. Hence, $\phi \in R(G)$. □

4.1.3. **Lemma.** *The subspace $R(G)$ is invariant for $R(f)$.*

Proof. Let $\phi \in R(G)$. By 2.3.5 we have

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)\,d\mu(h) = \sum_{i=1}^n a_i(g)\int_G f(h)b_i(h)\,d\mu(h)$$

for any $g \in G$, i.e., $R(f)\phi$ is a linear combination of a_i, $1 \le i \le n$. □

Let $E = R(G)^\perp$ in $L^2(G)$. Then, by 4.1.3, $R(G)$ is invariant for selfadjoint operator $R(f^*)R(f)$. This in turn implies that E is also invariant for this operator. Therefore the restriction of this operator to E is a positive selfadjoint compact operator. Assume that its norm is greater than 0. Then, by 3.2.1, the norm is an eigenvalue of this operator, and there exists a nonzero eigenvector $\phi \in E$ for that eigenvalue. Clearly, ϕ is an eigenvector for $R(f^*)R(f)$ too. By 4.1.2, ϕ is also in $R(G)$. Hence, we have $\|\phi\|_2^2 = (\phi \mid \phi) = 0$, and $\phi = 0$ in $L^2(G)$. Hence, we have a contradiction.

Therefore, the operator $R(f^*)R(f)$ is 0 when restricted to E. Hence

$$0 = (R(f^*)R(f)\psi|\psi) = \|R(f)\psi\|_2^2$$

for any $\psi \in E$. It follows that $R(f)\psi = 0$. Since $R(f)\psi$ is a continuous function, we have

$$0 = (R(f)\psi)(1) = \int_G f(g)\psi(g)d\mu(g),$$

i.e., ψ is orthogonal to $\bar{f}$.

Since $f \in C(G)$ was arbitrary and $C(G)$ is dense in $L^2(G)$, it follows that $\psi = 0$. This implies that $E = \{0\}$.

This completes the proof of the following result.

4.1.4. **Theorem** (Peter-Weyl)**.** *The algebra $R(G)$ is dense in $L^2(G)$.*

4.2. **Continuous version.** Let $g \in G$. Assume that $g \neq 1$. Then there exists an open neighborhood U of 1 such that U and Ug are disjoint. There exists positive function ϕ in $C(G)$ such that $\phi|_U = 0$ and $\phi|_{Ug} = 1$. This implies that

$$\begin{aligned}\|R(g)\phi - \phi\|^2 &= \int_G |\phi(hg) - \phi(h)|^2 d\mu(h) \\ &= \int_U |\phi(hg) - \phi(h)|^2 d\mu(h) + \int_{G-U} |\phi(hg) - \phi(h)|^2 d\mu(h) \geq \mu(U).\end{aligned}$$

Therefore $R(g) \neq I$. Since by 4.1.4, $R(G)$ is dense in $L^2(G)$, $R(g)|_{R(G)}$ is not the identity operator.

This implies the following result.

4.2.1. **Lemma.** *Let $g, g' \in G$ and $g \neq g'$. Then there exists a function $\phi \in R(G)$ such that $\phi(g) \neq \phi(g')$.*

Proof. Let $h = g^{-1}g' \neq 1$. Then there exists $\psi \in R(G)$ such that $R(h)\psi \neq \psi$. Hence, we have $R(g)\psi \neq R(g')\psi$. It follows that $\psi(hg) \neq \psi(hg')$ for some $h \in G$. Therefore, the function $\phi = L(h^{-1})\psi$ has the required property. □

In other words, $R(G)$ separates points in G. By Stone-Weierstrass theorem, we have the following result which is a continuous version of Peter-Weyl theorem.

4.2.2. **Theorem** (Peter-Weyl)**.** *The algebra $R(G)$ is dense in $C(G)$.*

Another consequence of 4.2.1 is the following result.

4.2.3. **Lemma.** *Let U be an open neighborhood of 1 in G. Then there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$.*

Proof. The complement $G - U$ of U is a compact set. Since $R(G)$ separates the points of G, for any $g \in G - U$ there exists a function $\phi_g \in R(G)$ and an open neighborhood U_g of g such that $|\phi_g(h) - \phi_g(1)| > \epsilon$ for $h \in U_g$. Since $G - U$ is compact, there exists a finite set $g_1, g_2, \cdots, g_m$ in $G - U$ such that $U_{g_1}, U_{g_2}, \cdots, U_{g_m}$ form an open cover of $G - U$ and $|\phi_{g_i}(h) - \phi_{g_i}(1)| > \epsilon$ for $h \in U_{g_i}$. Let π_i be a finite-dimensional representation of G with matrix coefficient ϕ_{g_i}. Then $\pi_i(h) \neq I$ for $h \in U_{g_i}$, $1 \leq i \leq n$. Let π be the direct sum of π_i. Then $\pi(h) \neq I$ for $h \in G - U$, i.e., $\ker \pi \subset U$. □

4.3. **Matrix groups.** Let G be a topological group. We say that G *has no small subgroups* if there exists a neighborhood U of $1 \in G$ such that any subgroup of G contained in U is trivial.

4.3.1. **Lemma.** *Let V be a finite-dimensional complex vector space. Then the group* $\mathrm{GL}(V)$ *has no small subgroups.*

Proof. Let $\mathcal{L}(V)$ be the space of all linear endomorphisms of V. Then $\exp : \mathcal{L}(V) \longrightarrow \mathrm{GL}(V)$ given by

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

defines a holomorphic map. Its differential at 0 is the identity map I on $\mathcal{L}(V)$. Hence, by the inverse function theorem, it is a local diffeomorphism.

Let U be an open neighborhood of 1 in $\mathrm{GL}(V)$ and V the open ball around 0 in $\mathcal{L}(V)$ of radius ϵ (with respect to the linear operator norm) such that $\exp : V \longrightarrow U$ is a diffeomorphism. Let V' be the open ball of radius $\frac{\epsilon}{2}$ around 0 in $\mathcal{L}(V)$. Then $U' = \exp(V')$ is an open neighborhood of 1 in $\mathrm{GL}(V)$. Let H be a subgroup of $\mathrm{GL}(V)$ contained in U'. Let $S \in H$. Then $S = \exp(T)$ for some $T \in V'$. Hence, we have $S^2 = \exp(T)^2 = \exp(2T) \in H$. Moreover, $S^2 \in H$ and $S^2 = \exp(T')$ for some $T' \in V'$. It follows that $\exp(T') = \exp(2T)$ for $2T, T' \in V$. Since exp is injective on V, we must have $2T = T'$. Hence, $T \in \frac{1}{2}V'$. It follows that $H \subset \exp\left(\frac{1}{2}V'\right)$. By induction we get that $H \subset \exp\left(\frac{1}{2^n}V'\right)$ for any $n \in \mathbb{N}$. This implies that $H = \{1\}$. □

A compact subgroup of $\mathrm{GL}(V)$ we call a compact *matrix* group.

4.3.2. **Theorem.** *Let G be a compact group. Then the following conditions are equivalent:*

(i) *G has no small subgroups;*
(ii) *G is isomorphic to a compact matrix group.*

Proof. (i)⇒(ii) Let U be an open neighborhood of $1 \in G$ such that it contains no nontrivial subgroups of G. By 4.2.3, there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$. This clearly implies that $\ker \pi = \{1\}$, and $\pi : G \longrightarrow \mathrm{GL}(V)$ is an injective homomorphism. Since G is compact, π is homoeomprphism of G onto $\pi(G)$. Therefore, G is isomorphic to the compact subgroup $\pi(G)$ of $\mathrm{GL}(V)$.

(ii)⇒(i) Assume that G is a compact subgroup of $\mathrm{GL}(V)$. By 4.3.1, there exists an open neighborhood U of 1 in $\mathrm{GL}(V)$ such that it contains no nontrivial subgroups. This implies that $G \cap U$ contains on nontrivial subgroups of G. □

4.3.3. **Remark.** For a compact matrix group G, since matrix coefficients of the natural representation separate points in G, 4.2.1 obviously holds. Therefore, in this situation, Stone-Weierstrass theorem immediately implies the second version of Peter-Weyl theorem, which in turn implies the first one.

4.3.4. **Remark.** By Cartan's theorem [**?**], any compact matrix group is a Lie group. On the other hand, by [**?**] any Lie group has no small subgroups. Hence, compact Lie groups have no small subgroups and therefore they are compact matrix groups.

4.3.5. **Remark.** Let $T = \mathbb{R}/\mathbb{Z}$. Then T is a compact abelian group. Let G be the product of inifinite number of copies of T. Then G is a compact abelian group. By the definition of product topology, any neghborhood of 1 contains a nontrivial subgroup.

Let G be an arbitrary compact group. Let (π, V) be a finite-dimensional representation. Put $N = \ker \pi$. Then N is a compact normal subgroup of G and G/N equipped with the quotient topology is a compact group. Clearly, G/N is a compact matrix group.

Let $\mathcal{S}$ be the family of all compact normal subgroups N of G such that G/N is a compact matrix group. Clearly, N, N' in $\mathcal{S}$ implies $N \cap N' \in \mathcal{S}$. Therefore, $\mathcal{S}$ ordered by inclusion is a directed set. One can show that G is a projective limit of the system G/N, $N \in \mathcal{S}$. Therefore, any compact group is a projective limit of compact matrix groups. By the above remark, this implies that any compact group is a projective limit of compact Lie groups.