Consequences of Peter-Weyl theorem

Remark. Uniform version of Peter-Weyl is stronger than the $L^2$-version (as discussed last time). The only place in the proof of uniform version where we used $L^2$-version is to show that $R(G)$ differs points in $G$.

If $G$ is a linear group (i.e., a group of matrices like $SO(n)$) the natural
representation differs points of $G$ and Peter-Weyl theorem follows immediately from Stone-Weierstrass theorem. We shall discuss this later in more detail.

Since $R(G)$ differs points of $G$, for any $g \in G, g \neq 1$, there exists $f \in R(G)$ such that $f(g) \neq f(1)$.

Since $f$ is a matrix coefficient of a finite-dimensional representation, we get
Lemma. Let \( g \in G, g \neq 1 \).
Then there exists a finite-dimensional representation \((\pi, \mathbb{V})\) of \( G \) such that \( \pi(g) \neq I \).

Now we prove a variant of a result we proved for finite groups.
Let \((\pi, \mathbb{V})\) be a finite-dimensional representation of \( G \). Let \( \langle \cdot, \cdot \rangle \) be an arbitrary inner product on \( \mathbb{V} \), since \( \pi \) is continuous for any \( u, v \in \mathbb{V} \) the function
\[ g \mapsto \langle \pi(g)u, \pi(g)v \rangle \]

is continuous. Hence
\[
(\mu|\nu) = \int \langle \pi(g)u, \pi(g)v \rangle \, d\mu(g)
\]

is well-defined function on \( V \times V \).

As in the case of finite groups we show directly that \((.1,.)\) is linear in first and antilinear in second variable.

Moreover, \( (\mu|\nu) = (\nu|\mu) \) and \( (\mu|\nu) \geq 0 \) for \( u, v \in V \)

Moreover, we have
\[(u|u) = \int_G <\pi(g)u|\pi(g)u> \, dm(g),\]

\[<\pi(g)u|\pi(g)u> \geq 0 \text{ for all } g \in G, \text{ and}\]

\[g \mapsto <\pi(g)u|\pi(g)u> \text{ is continuous}.\]

Assume that \( u \neq 0 \).

Then \( <u|u> > 0 \). Hence there exist an open neighborhood \( \mathcal{U} \) of 1 such that

\[<\pi(g)u|\pi(g)u> > \varepsilon\]

on \( \mathcal{U} \) for some \( \varepsilon > 0 \).

Hence

\[(u|u) = \int_G <\pi(g)u|\pi(g)u> \, dm(g) = \]

\[= \int_G <\pi(g)u|\pi(g)u> \, dm(g).\]
\[
\begin{align*}
= & \int \langle \pi(g) u | \pi(g) u \rangle \, d\mu(g) + \\
\geq & \varepsilon \cdot \mu(u).
\end{align*}
\]

Since \( \mu(u) > 0 \) it follows that \( (u | u) > 0 \).

It follows that \( (u | u) = 0 \) implies \( u = 0 \) and C.1. is an inner product on \( V \).

Moreover

\[
\langle \pi(g)u | \pi(g)u \rangle = \\
= \int \langle \pi(h) \pi(g) u | \pi(h) \pi(g) u \rangle \, d\mu(h)
\]
\[
\int_G \langle \pi(h) u | \pi(h) v \rangle \, d\mu(h) = 1
\]

= \int_G \langle \pi(h) u | \pi(h) v \rangle \, d\mu(h) = (u|v)

by the invariance of Haar measure.

Hence, \( \pi(h) \) are unitary operators with respect to (1.1), and \( \pi \) is a unitary representation. This proves the following result.

**Proposition** Any finite-dimensional representation of \( G \) is unitary (with respect to an appropriate...
inner product.

Let \((\pi, V)\) be a finite-dimensional representation of \(G\) (unitary with respect to \((I, 1)\)). Let \(U\) be a \(G\)-invariant subspace of \(V\). Then \(U\) is invariant for all \(\pi(g), g \in G\).

Hence \(U^\perp\) is invariant for all \(\pi(g^\star), g \in G\).

Since \(\pi\) is unitary, \(\pi(g^\star) = \pi(g^{-1})\), so \(U^\perp\) is \(G\)-invariant.

Hence, \(V = U \oplus U^\perp\) is
a direct sum of representations.

By induction in dimension of $\pi$ we prove the following theorem.

**Theorem.** Every finite-dimensional representation of $G$ is a direct sum of irreducible representations.

A direct consequence is the following result.

**Theorem.** Let $g \in G$, $g \neq 1$.

Then there exists an irreducible finite-dimensional representation $(\pi, V)$ of $G$ such that $\pi(g) \neq I$. 
As we remarked, the irreducible representation in above theorem is unitary (with respect to appropriate inner product).

Remark: Let $(\pi, V)$ be a representation of a locally compact group $G$ on a Hilbert space $V$. We say that $\pi$ is irreducible if the only closed $G$-invariant subspaces of $V$ are $\{0\}$ and $V$.

If $V$ is finite-dimensional
this agrees with old definition (since all finite-dim. subspaces are closed).

We have the following generalization of the above result:

**Theorem (Gelfand–Raikov)**

Let $G$ be a locally compact group. Let $g \in G$, $g \neq 1$. Then there exists an irreducible unitary representation $(\pi, V)$ of $G$ such that $\pi(g) \neq I$. 
The proof of Gelfand-Raikov theorem is quite different from our discussion. The essential point of the theorem is that we have to consider infinite dimensional irreducible unitary representations of $G$. Finite-dimensional irreducible unitary representations do not suffice!

This can be seen from the following theorem.
Theorem (Segal-von Neumann)
Let $G$ be a noncompact simple Lie group. Let $(\pi, V)$ be a finite-dimensional irreducible unitary representation of $G$. Then $(\pi, V)$ is trivial, (i.e., $V$ is one-dim., and $\pi(g) = 1$, for all $g \in G$).

Remark: $SL(n, \mathbb{R})$, are examples of noncompact simple Lie groups.