Proof of Peter-Weyl ($L^2$-version)

Last time we constructed the linear operator $R(f)$ on $L^2(G)$ and proved that it is compact for any $f \in C(G)$.

Now we are going to use its properties to prove the following version of Peter-Weyl theorem.

**Theorem.** The subspace $R(G)$ is dense in $L^2(G)$.

Let $\varphi \in R(G)$. Then $\varphi$ is a matrix coefficient of a finite-dimensional representation of $G$. 

Denote this representation by $(\gamma, U)$. Then, if $n = \dim U$, $\gamma$ is a linear combination of matrix coefficients $\gamma(g)_{ij}$ in some basis of $U$. Moreover, we have

$$v(gh)_{ij} = \sum_{k=1}^{m} \gamma(g)_{ik} \gamma(h)_{kj}.$$ 

This implies that

$$y(gh) = \sum_{\ell=1}^{m} a_{\ell}(g) b_{\ell}(h)$$

for some functions $a_{\ell}, b_{\ell} \in \mathbb{R}(G)$. This is a stronger form of 2.3.2 in texed notes.
Let \( f \in C(G) \) and \( \varphi \in \mathcal{R}(G) \).

Then
\[
(Rf)(g) = \int_{G} f(h) \varphi(gh) \, d\mu(h)
\]
\[
= \int_{G} \sum_{\ell=1}^{m} a_{\ell}(g) b_{\ell}(h) \, d\mu(h)
\]
\[
= \sum_{\ell=1}^{m} \left( \int_{G} f(h) b_{\ell}(h) \, d\mu(h) \right) a_{\ell}(g)
\]
\[
= \sum_{\ell=1}^{m} c_{\ell} a_{\ell}(g)
\]
\[
\Rightarrow \quad (Rf) \varphi \in \mathcal{R}(G).
\]

Hence,
\[
(Rf)(\mathcal{R}(G)) \subset \mathcal{R}(G).
\]

Since \( Rf \) is continuous,
\[
(Rf)(\overline{\mathcal{R}(G)}) \subset \overline{\mathcal{R}(G)}.
\]
Let \( E = (R(G))^\perp = (\overline{R(G)})^\perp \)

Then \( E \) is a closed subspace of \( L^2(G) \). Moreover, we have

\[
L^2(G) = \overline{R(G)} \oplus E
\]
as orthogonal sum of two closed subspaces.

Let \( \psi \in E \) and \( \varphi \in \overline{R(G)} \).

Then

\[
(R(f)\varphi | \psi) = (\varphi | R(f)^* \psi) = \overline{(\varphi | R(f^*) \psi)} = 0
\]
Hence $R(f)\varphi \in E$, i.e. $E$ is $R(f)$-invariant.

**Claim:** $R(f): E \rightarrow E$ is zero for all $f \in C(G)$.

Assuming the claim we are going to prove the theorem.

Let $\varphi \in E$. Then $R(f)\varphi$ is a continuous function and

$$\left( R(f)\varphi \right) (g) = \int_G f(h) \varphi(gh) d\mu(h)$$

If $R(f) = 0$ on $E$, $R(f)\varphi = 0$ in $E$ and $(R(f)\varphi)(i) = 0$ since $R(f)\varphi$ is continuous.
Therefore
\[ \int_{\mathbb{R}} f(x) \varphi(x) \, dx = 0, \]
i.e. \( (f, \varphi) = 0 \).

Therefore \( \varphi \perp C(G) \) is \( L^2(G) \). Since \( C(G) \) is dense in \( L^2(G) \), \( \varphi \perp L^2(G) \),
\[ \implies \varphi \perp \varphi \implies \varphi = 0 \implies \varphi = 0. \]

This proves that \( E = \{0\} \) and \( L^2(G) = \overline{R(G)} \),
what is the statement of the theorem.
It remains to prove the claim.

Assume that \( f \in C(G) \) is such that \( R(f)|_E \) is not zero.

We have

\[
(R(f)^* R(f) \varphi | \varphi) = (R(f) \varphi | R(f) \varphi) = |R(f) \varphi|^2.
\]

This implies that

\( R(f)^* R(f) = R(f^*) R(f) : \)

\( E \to E \) is not zero.

From the last lecture, we know that it is a
positive selfadjoint compact operator),
If it is nonzero, from its spectral theory we know that it must have an eigenvalue \( \lambda > 0 \) with finite-dimensional eigenspace \( F \).
Let \( \varphi \in F \). Then
\[
R(f)^* R(f) \varphi = \lambda \varphi
\]
and \( R(f) \varphi \) is a continuous function. Hence \( \varphi \) is continuous, i.e., it is in \( C(G) \).
Now

\[(R(f)L(g)\varphi)(h) = \int h \cdot \varphi(k) (L(g)\varphi)(h-k) \, d\mu(k) = \int \varphi(k) \varphi(g^{-1}h-k) \, d\mu(k) = (R(f)\varphi)(g^{-1}k) = (L(g)R(f)\varphi)(h) \\Rightarrow \]

\[R(f)L(g)\varphi = L(g)R(f)\varphi.\]

Hence

\[(R(f)^*R(f))L(g)\varphi = \]

\[= R(f)^*L(g)R(f)\varphi = \]

\[= R(f^*)L(g)R(f)\varphi = \]

\[= L(g)R(f^*)R(f)\varphi = \]

\[= L(g)(R(f)^*R(f))\varphi = \lambda L(g)\varphi.\]
Hence, \( L(g)\psi \in F \), for any \( g \in G \).

It follows that \( \psi \) is \( G \)-finite, i.e., \( \psi \in \mathcal{R}(G) \).

But \( \psi \in E \), so \( \psi \perp \psi \)

\[ \Rightarrow (\psi | \psi) = 0 \Rightarrow \| \psi \|^2 = 0 \]

\[ \Rightarrow \psi = 0! \]

Hence any eigenfunction in \( F \) is 0 and we have a contradiction.

This proves the claim and completes the proof of \( L^2 \)-version of Peter-Weyl theorem.