Let $G$ be a compact group. To determine $G$ it is enough to find all irreducible characters $\text{ch}(\pi), \pi \in \hat{G}$.

Assume that $G$ is connected. As we remarked, any finite-dimensional representation $\pi$ is actually a representation of the group $G/\ker\pi$ which is a connected compact Lie group.

We shall use some structural results about connected compact Lie groups proved.
in Lie Groups class.

Let $G$ be a connected compact Lie group. A torus $T$ in $G$ is a closed subgroup isomorphic to a product of circle groups. Let $T$ be the family of tori in $G$ ordered by inclusion. Then $T$ contains maximal elements. They are all conjugate by elements of $G$. Moreover, any torus is contained in a maximal torus. Any element $g \in G$ is contained in a maximal
Fix a maximal torus $T$ in $G$. Then the conjugation map

$$G \times T \rightarrow G
\quad (g, t) \mapsto g^t g^{-1}$$

is surjective. Since characters are constant on conjugacy classes of $G$, they are completely determined by their restriction to $T$.

**Example:** Let $G = \text{SU}(2)$ - $2 \times 2$ unitary matrices of determinant equal to 1.
Therefore, the elements of SU(2) are of the form

$$\begin{pmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Therefore SU(2) is diffeomorphic to a 3-dimensional sphere and simply connected.

The eigenvalues $\lambda$ such matrix are complex numbers of modulus 1. If $\lambda_1$ and $\lambda_2$ are its eigenvalues, $\lambda_1 \cdot \lambda_2 = 1$, i.e. the eigenvalues are $e^{i\phi}$ and $e^{-i\phi}$.
Let \( T = \{ (e^{i\phi} 0, 0 e^{-i\phi}) ; \phi \in \mathbb{R} \} \).

Then \( T \) is a torus in \( G \).

One can show that \( T \) is a maximal torus in \( G \).

Let \( g \in G \) with eigenvalues \( e^{i\phi} \) and \( e^{-i\phi} \). Then there exists an orthonormal basis \( \{ v_1, v_2 \} \) of \( \mathbb{C}^2 \) consisting of eigenvectors of \( g \). The matrix having \( v_1 \) and \( v_2 \) as columns is unitary matrix which conjugates \( g \) with
\((e^{i\varphi} 0\ 0\ e^{-i\varphi})\). By dividing it by a square root of the determinant, we can assume that this matrix is \(h \in SU(2)\).

This proves that \(G \times T \to G\) is surjective in this case.

Since \(\pi|_T\) is a finite-dimensional of \(T\), it is a direct sum of characters. Therefore

\[
\text{ch}(\pi)|_T = \sum_{\omega \in \hat{T}} m_{\omega} \omega
\]

where \(\hat{T}\) is a group of characters of \(T\) (which is isomorphic to \(\mathbb{Z}^r\), as we
discussed before). The number \( \tau \) is the dimension of the torus \( T \) - the rank of \( G \).

\( m_\nu \in \mathbb{Z}_+ \) is the multiplicity of \( \nu \in \hat{T} \) in \( \pi \).

\( \nu \in \hat{T} \) is a weight of \( \pi \) if \( m_\nu \neq 0 \).

Therefore, to find characters of reducible representations of \( G \) we need to determine the multiplicities of their weights.

We shall explain a method
to do this, due to Hermann Weyl, based on orthogonality relations.
We shall just state various results in general. The detailed proofs for SU(2) are on the texed note "Weyl character formula" on the class web page.
The first step is the Weyl integral formula. It gives the formula for the integral
over \( G \) as the integral over conjugacy classes followed by integration over \( T \). This is the Weyl integral formula.

First, since \( T \) is abelian we can factor the map

\[
\begin{array}{ccc}
G \times T & \rightarrow & G \\
\downarrow & & \uparrow p \\
G/_{\mathcal{H}} \times T & & \\
\end{array}
\]

In this way,

\[
\dim (G/_{\mathcal{H}} \times T) = \dim (G/_{\mathcal{H}}) + \dim T = \dim G - \dim T + \dim T = \dim G.
\]
Let $N(T) = \{ n \in G \mid m T n^{-1} = T \}$ be the normalizers of $T$.

Then $N(T)$ is a closed subgroup of $G$, $T$ is its identity component and

$$W = N(T) / T$$

is the Weyl group of $(G, T)$.

There exists an open dense subset $G_{\text{reg}}$ of $G$ (such that $G \setminus G_{\text{reg}}$ is of Haar measure zero) such that

$$p^{-1}(G_{\text{reg}}) = G / T \times T_{\text{reg}}$$

($T_{\text{reg}} = T \cap G_{\text{reg}}$) is a $|W|$-fold cover of $G_{\text{reg}}$. 
G acts on $G/T$ by left multiplication. There exists a unique positive \( G \)-invariant measure \( \nu \) on $G/T$ which is normalized, i.e., \( \nu(G/T) = 1 \). Then we have

$$
\int_G f(g) \, d\mu_G(g) = \\
\frac{1}{[W]} \int_{G/T} \left( \int_{G/T} f(g') \delta(g^{-1}) \, d\nu(g) \right) \\
D(t) \, d\mu_T(t)
$$

where \( D \) is the absolute value of Jacobian of \( p: G/T \times T \rightarrow G \).
In the texed note, the function $D$ is calculated explicitly for $SU(2)$. General case requires details on the structure of Lie algebra of $G$. In the next installment we shall describe the function $D$ in general and discuss Weyl character formula.