Let $G$ be a Lie group.

$\lambda \in \wedge^n T^*_1(G), \quad n = \dim G$

$\omega_g = \wedge^n T_g (\delta(g^{-1}))^* \lambda$

$\omega$ is left invariant $n$-form on $G$.

$\rightarrow$ corresponding positive measure $\mu = |\omega|$ is a left Haar measure on $G$.

This completes the proof of existence.

$G$ - Lie group, $\mu$ - left Haar measure

$\lambda : f \rightarrow \int_G f(gh) \, d\mu(g)$ a left invariant measure
\[ \Delta(h \in G \int f(hg) \, d\mu(g)) = \int f(g) \, d\mu(g) \]

\[ \Delta(h) > 0 \quad \Delta : G \rightarrow \mathbb{R}_+^* \]

\(-\text{continuous (exercise)}\)

\[ \Delta(g_1 g_2) \int f(hg_1 g_2) \, d\mu(h) = \]

\[ = \Delta(g_1) \int f(h g_2) \, d\mu(h) = \]

\[ = \Delta(g_1) \Delta(g_2) \int f(h) \, d\mu(h) \]

\[ \Rightarrow \quad \Delta(g_1 g_2) = \Delta(g_1) \cdot \Delta(g_2) \]

\(-\text{is a group homomorphism}\)

\(-\text{is a Lie group homomorphism}\)

\(-\text{Modular \( \Delta \) : \( G \rightarrow \mathbb{R}_+^* \)}\)

\(-\text{function of group of positive reals}\)
G is unimodular if
\[ \Delta_G = 1. \]

**Theorem.** Let G be a compact Lie group. Then G is unimodular.

**Proof.**
\[
\Delta(g) \int_G 1(hg) \, dh = \Delta(g) \cdot \mu(G)
\]

\[
\int_G 1(h) \, dh = \mu(G)
\]

**Exercise.** G is a two-dimensional connected non-abelian Lie group \( \Rightarrow G \) is not unimodular!
If a Lie group $G$ is unimodular and $\mu$ a left Haar measure on $G$,

$$\int f(g) \, d\mu(g) = \int f(gh) \, d\mu(g)$$

$$= \int f(hg) \, d\mu(g)$$

$\Rightarrow \mu$ is right invariant, i.e., $\mu$ is bi-invariant – Haar measure on $G$.

Assume that $G$ is a compact Lie group, $\mu$ a Haar measure on $G$. Then $\mu(G) > 0$.

By replacing $\mu$ with its multiple, we can assume that
\[
\mu(G) = 1. \text{ Such Haar measure is unique.}
\]

- normalized Haar measure on \( G \).

**Theorem.** Let \( G \) be a Lie group. Then the following conditions are equivalent:

(i) \( G \) is compact;

(ii) \( \mu(G) \) is finite.

**Proof.** We proved (i) \( \Rightarrow \) (ii).

Assume that \( \mu(G) \) is finite. Let \( V \) be a compact neighborhood of \( 1 \). Then \( \mu(V) > 0 \).
Denote by $\mathcal{F}$ the family of finite sets $\{g_1, g_2, \ldots, g_m\}$ such that $g_i V \cap g_j V = \emptyset$ for all $i \neq j$, $1 \leq i, j \leq m$.

Then
$$\mu\left(\bigcup_{i=1}^{m} g_i V\right) = m \cdot \mu(V) \leq \mu(G).$$

Hence
$$m \leq \frac{\mu(G)}{\mu(V)}$$
i.e. $m$ is bounded.

Let $m$ be maximal possible.

Then for $\{g_1, \ldots, g_m\}$ in $\mathcal{F}$, and $g \in G$ we have
$$gV \cap g_i V \neq \emptyset$$ for some $i.$
This implies
\[ g \subseteq g_iVV^{-1} \]
\[ \implies G = \bigcup_{i=1}^{\infty} g_iVV^{-1} \]

Since \( V \) is compact, \( V^{-1} \) is also compact \( \implies \) \( VV^{-1} \) is compact \( \implies g_iVV^{-1} \) is compact
\[ \implies G \) is compact. \]
Invariant inner product on $L(G)$, $G$ compact.

Let $G$ be a compact Lie group. Take an arbitrary inner product $(\cdot, \cdot)$ on $L(G)$.

$$g \mapsto (\text{Ad}(g) \xi, \text{Ad}(g) \eta)$$

is a continuous function on $G$.

Put

$$\langle \xi, \eta \rangle = \int_G (\text{Ad}(g) \xi, \text{Ad}(g) \eta) \, d\mu(g)$$

where $\mu$ is the normalized Haar measure on $G$. 

$\langle \cdot, \cdot \rangle : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{R}$ is a bilinear form.

Since $(\cdot, \cdot)$ is symmetric, $\langle \cdot, \cdot \rangle$ is also symmetric.

$$\langle \xi, \xi \rangle = \int_G (\text{Ad}(g)\xi, \text{Ad}(g)\xi) \, d\mu(g) =$$

$$= \int_G \|\text{Ad}(g)\xi\|^2 \, d\mu(g) \geq 0$$

continuous positive

$\langle \xi, \xi \rangle = 0$ implies that

$$\int_G \|\text{Ad}(g)\xi\|^2 \, d\mu(g) = 0$$

Assume that $\xi \neq 0$. Then
\[ \| \xi \| > 0. \] Therefore, there exists an open neigh-
bor \( U \) of \( 1 \) such that
\[ \| \text{Ad}(g) \xi \| \geq \frac{1}{2} \| \xi \| \]
for \( g \in U \).

\[ \Rightarrow \sum_{g} \| \text{Ad}(g) \xi \|^{2} \, d\mu(g) \geq \]
\[ \geq \sum_{U} \| \text{Ad}(g) \xi \|^{2} \, d\mu(g) \geq \frac{1}{4} \sum_{U} \| \xi \|^{2} \, d\mu(g) = \]
\[ = \frac{1}{4} \mu(U) \| \xi \|^{2} \]

Hence \( \langle \xi, \xi \rangle > 0 \) and \( \langle \cdot, \cdot \rangle \) is an inner product
on \( L(G) \).
\[ \langle \text{Ad}(g) \xi, \text{Ad}(g) \eta \rangle = \]

\[ = \int_G \langle \text{Ad}(h) \text{Ad}(g) \xi, \text{Ad}(h) \text{Ad}(g) \eta \rangle \, d\mu(h) \]

\[ = \int_G \langle \text{Ad}(hg) \xi, \text{Ad}(hg) \eta \rangle \, d\mu(h) = \]

\[ = \int_G \langle \text{Ad}(h) \xi, \text{Ad}(h) \eta \rangle \, d\mu(h) = \]

\[ = \langle \xi, \eta \rangle. \]

Hence \( \langle \cdot, \cdot \rangle \) satisfies

\[ \langle \text{Ad}(g) \xi, \text{Ad}(g) \eta \rangle = \langle \xi, \eta \rangle \]

for any \( g \in G, \xi, \eta \in L^2(G) \).

Hence it is an \( G \)-invariant inner product.
By differentiation we get

\[ \langle \text{ad}(\xi) \xi, \eta \rangle + \langle \xi, \text{ad}(\xi) \eta \rangle = 0 \]

for all \( \xi, \eta, \zeta \in L(G) \).

\[ \Rightarrow \text{ad}(\xi), \xi \in L(G), \text{ is } \]

antisymmetric linear map with respect to the invariant inner product.

The existence of invariant inner product on \( L(G) \) allows to say a lot on its structure.
Let \( \mathfrak{o} \) be an ideal in \( L(G) \).
Let \( \mathfrak{o}^\perp \) be the orthogonal complement of \( \mathfrak{o} \)
\( \xi \in \mathfrak{o}^\perp, \eta \in \mathfrak{o} \implies \langle \xi, \eta \rangle = 0 \)
Let \( \mathfrak{s} \in L(G) \). Then \( [\xi, \eta] \in \mathfrak{o} \)
\( \implies 0 = \langle \xi, (\text{ad}_\mathfrak{s})(\eta) \rangle = -\langle (\text{ad}_\mathfrak{s})(\xi), \eta \rangle \)
\( = -\langle [\xi, \xi], \eta \rangle \)
\( \implies [\xi, \xi] \in \mathfrak{o}^\perp \).
\( \mathfrak{o}^\perp \) is an ideal.
\( L(G) = \mathfrak{o} \oplus \mathfrak{o}^\perp \) as linear spaces,
\( \xi \in \mathfrak{o}, \eta \in \mathfrak{o}^\perp \implies [\xi, \eta] \in \mathfrak{o} \cap \mathfrak{o}^\perp = \{0\} \).
\( L(G) \) is a direct sum of
two ideals.

By induction

$L(G)$ is a direct sum of minimal ideals.

$M$ - minimal ideal in $L(G)$.

$\exists M$ ideal in $M$

$L(G) = M \oplus M^\perp \Rightarrow$

$B$ is an ideal in $L(G)$.

$\Rightarrow B = \{0\}$ or $B = M$.

There are two options

1. $\dim M = 1$, $M$ is abelian

   $\xi \in M$, $\text{ad} \xi |_M = 0$, $\text{ad} \xi |_{M^\perp} = 0$

   $\text{ad} \xi = 0 \Rightarrow \xi \in Z - \text{center of } L(G)$

   $m \subset Z$. 

② $M$ is not abelian.

Then $\dim M > 1$ and $M$ has no nontrivial ideals.

- simple Lie algebra.

$L(G)$ is a direct sum of its center $Z$ and simple ideals.

Example ① of two dimensional Lie algebra with basis $e_1, e_2$ and $[e_1, e_2] = e_1$.

Then $R(e, 1)$ is an ideal in $M$.

$[1, 1] = 0$ in not simple.
This implies that the dimension of a simple Lie algebra \( \geq 3 \).

\[ \mathfrak{q}_1 = L(\text{SU}(2)) \]

\[ \mathfrak{q}_1 = \{ \left( \begin{array}{cc} ix & y+iz \\ y-iz & -ix \end{array} \right) ; x, y, z \in \mathbb{R} \} \]

\( \mathfrak{q}_1 \) is simple.