Let $O_f$ be a Lie algebra.

Denote by $R$ the family of all solvable ideals in $O_f$. If $\mathcal{F} \subseteq R$, then $R$ is not empty.

If $\mathfrak{a}, \mathfrak{b} \in R$, then $\mathfrak{a} + \mathfrak{b}$ is also an ideal.

$$\mathfrak{a} + \mathfrak{b} \cong \mathfrak{a} / \mathfrak{a} \cap \mathfrak{b}$$

Since $\mathfrak{a}$ is solvable, $\mathfrak{a} / \mathfrak{a} \cap \mathfrak{b}$ is also solvable. Hence $$(\mathfrak{a} + \mathfrak{b}) / \mathfrak{b}$$ is solvable.

Since $\mathfrak{b}$ is solvable, $\mathfrak{a} + \mathfrak{b}$ is solvable.
Therefore \( a + b \in R \).

If we equip \( R \) with partial order given by inclusion, any element of \( R \) is in a maximal element, because of finite-dimensionality.

If \( M \) and \( N \) are maximal solvable ideals, \( M + N \) is a solvable ideal. By maximality, we see that \( M = M + N = N \).

\( R \) contains largest element \( r \).

The radical \( r \) of \( \mathfrak{g} \)

\( \mathfrak{g} \) is semisimple Lie algebra if its radical \( r = \{0\} \).
Lemma. Equivalent:
① \( \mathfrak{g} \) is semisimple;
② \( \mathfrak{g} \) contains no nontrivial abelian ideals.

Proof: If \( \mathfrak{g} \) is semisimple, we have \( \mathfrak{z} = \{0\} \), all solvable ideals are \( 0 \). This implies ②.

② If \( \mathfrak{z} \) is radical, \( D^n + \neq 0, D^{n+1} = 0 \), \( D^n + \) is abelian. \( \Box \)

Corollary: If \( \mathfrak{g} \) is semisimple, its center is \( \{0\} \).

A Lie group \( G \) is semisimple if \( \mathcal{L}(G) \) is semisimple.

A compact Lie group is semisimple if and only if the center of \( \mathcal{L}(G) \) is \( \{0\} \).
Proof. $L(G)$ is semisimple

$\Rightarrow$ center in $\mathfrak{so}_3$.

Assume the center in $\mathfrak{so}_3$. If $\mathfrak{o}_3$ is not semisimple, it contains a nonzero abelian ideal $\mathfrak{z}$

$L(G) = \mathfrak{o}_3 \oplus \mathfrak{o}_3$

$\mathfrak{o}_3$ is in the center of $L(G)$

$\Rightarrow \mathfrak{o}_3 = \{0\}$, contradiction.

If $G$ is a connected compact Lie group. Equivalent:

@ $G$ is semisimple;

@ center $Z$ of $G$ is finite.
Theorem (H. Weyl)
Let $G$ be a connected compact Lie group. Equivalent:
(c) $G$ is semisimple
(a) $G$ is compact.

Let $G$ be a connected compact Lie group. Then there exists a finite cover of $G$ which is a product of connected compact semisimple Lie group $K$ and a torus $T$.

$\tilde{G} = K \times \mathbb{R}^P$

compact.
**Theorem.** Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{r} \) its radical. Then \( \mathfrak{g}/\mathfrak{r} \) is semisimple.

**Proof.** Let \( \mathfrak{a} \) be an solvable ideal in \( \mathfrak{g}/\mathfrak{r} \). Let \( \varphi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \) \( \varphi^{-1}(\mathfrak{r}) = \mathfrak{B} \) \( \mathfrak{B} \) is an ideal in \( \mathfrak{g} \)

\[
0 \to \mathfrak{r} \to \mathfrak{B} \to \mathfrak{a} \to 0
\]

\( \varphi \) solvable \( \mathfrak{B} \) solvable \( \mathfrak{a} \) solvable

\( \Rightarrow \mathfrak{B} = \mathfrak{r} \Rightarrow \mathfrak{a} = 0. \) \( \square \)
Killing form of finite-dimensional Lie algebra over a field $k$ of characteristic 0.

$$B(\xi, \eta) = tr(\text{ad}\xi \circ \text{ad}\eta)$$

bilinear form.

A an automorphism of $G$

$$\text{ad}(A\xi)(\eta) = [A\xi, \eta] = A[\xi, A^{-1}\eta] = (A \circ \text{ad}\xi \circ A^{-1})(\eta)$$

$$\text{ad}(A\xi) = A \circ \text{ad}\xi \circ A^{-1}$$

$$B(A\xi, A\eta) = tr(\text{ad}(A\xi) \circ \text{ad}(A\eta)) =$$

$$= tr(A \circ \text{ad}\xi \circ A^{-1} \circ A \circ \text{ad}\eta \circ A^{-1}) =$$

$$= B(\xi, \eta)$$
Killing form is $\text{Aut}(g)$-invariant.

$G$ is a Lie group, $g \in G$.

$\text{Ad}(g)$ is in $\text{Aut}(\mathfrak{L}(G))$.

$$B(\text{Ad}(g)\xi, \text{Ad}(g)\eta) = B(\xi, \eta).$$