Let $G$ be a connected compact Lie group.

$\tilde{G}$ - universal covering group of $G$

$\tilde{G} \rightarrow G$

$\ker p = C$ - central

$C$ - discrete subgroup

There are two options

① $C$ is finite

② $C$ is infinite

① $\Rightarrow \tilde{G}$ is compact

② $C$ is finitely generated

$C = T \times \mathbb{Z}^+$  $p > 0$

finite torsion group
There is a group homomorphism
\[ C = \mathbb{T} \times \mathbb{Z}^p \]
\[ \varphi \quad \text{epimorphism} \quad \mathbb{Z} \]
\[ \varphi \quad \text{extends to a Lie group morphism} \quad \Phi : \tilde{G} \to \mathbb{R} \]
\[ \Phi = 0. \]
Therefore, \( L(\Phi) : L(\tilde{G}) \to L(\mathbb{R}) \)
\[ L(\tilde{G}) \to \mathbb{R} \]
is nonzero.
Since the Lie algebra \( L(\mathbb{R}) \) is abelian, \( L(\Phi)([L(G),L(G)]) = 0. \)
Therefore we must have
\[ [L(G),L(G)] \neq L(G). \]
$[L(G), L(G)]$ is an ideal in $L(G)$.

\( \mathfrak{y} \) an ideal in $L(G)$.

\( \mathfrak{y}^\perp \) orthogonal complement with respect to an invariant inner product.

Lemma: (i) $\mathfrak{y}^\perp$ is an ideal in $L(G)$

(ii) $L(G) = \mathfrak{y} \oplus \mathfrak{y}^\perp$

(direct sum of ideals).

Proof. Let $\xi \in \mathfrak{y}^\perp$ and $\eta \in \mathfrak{y}$.

Then

$$ ([\xi, \xi], \eta) = - ([\xi, \xi], [\xi, \eta]) = 0 $$

for all $\xi \in L(G)$, $\eta \in \mathfrak{y}$.
$[\xi, \xi] \in \mathfrak{y}^\perp.$
Hence, $\mathfrak{y}^\perp$ is an ideal.

$[\xi, \eta] \in \mathfrak{y} \cap \mathfrak{y}^\perp = \mathfrak{z}$
$L(G) = \mathfrak{y} \oplus \mathfrak{y}^\perp. \quad \Box$

Hence, $[L(G), L(G)]^\perp$ is an ideal in $L(G)$.

**Lemma.** $[L(G), L(G)]^\perp$ is the center $\mathfrak{z}$ of $L(G)$.

**Proof.** Let $\xi, \eta \in L(G)$

$\forall \xi \in [L(G), L(G)]^\perp \iff (\xi \mid [\xi, \eta]) = 0$

for all $\xi, \eta \iff ([\xi, \xi] \mid \eta)$ for all $\xi, \eta$

$\iff [\xi, \xi] = 0$ for all $\xi$

$\iff \xi \in \mathfrak{z}. \quad \Box$
\[ Z \neq \{0\}. \]

This implies that the center \( Z \) of \( G \) has Lie algebra \( L(Z) = Z \). Hence, \( \dim Z > 0 \).

The identity component \( Z_0 \) of \( Z \) is a connected compact abelian Lie group. Hence \( Z_0 \) is a torus of dimension \( > 0 \).

Let \( K \) be the integral subgroup corresponding to \([L(G),L(G)]\).

Let \( \tilde{K} \) be the universal covering group of \( K \).
Consider the projection
\[ G \leftarrow K \]
The morphism
\[ \pi \] \quad \alpha \quad \frac{\alpha}{\pi} \]
has differential
\[ \frac{\alpha}{G/Z_0} \]
which is an isomorphism of Lie algebras. Therefore this is a covering map. Hence \( K \) is the universal covering of \( G/Z_0 \).

Clearly, \( G/Z_0 \) is compact Lie group.

\[ L(G/Z_0) = [L(G), L(G)] \]

Let \( z \) be in the center of \[ [L(G), L(G)] \]. Then it
commutes with \([L(G), L(G)]\)
and \(Z \Rightarrow \) commutes with \(L(G) \Rightarrow z \in Z\). Hence
\(y \in Z \cap [L(G), L(G)] = \{0\}\).

Hence the center of \(L(G/Z_0)\)
is trivial.

\(\Rightarrow [L(G/Z_0), L(G/Z_0)] = L(G/Z_0)\)
\(\Rightarrow\) the universal cover
\(\tilde{K}\) of \(G/Z_0\) is compact.

Hence, \(K\) is compact.

It follows that \(K\) is a
closed subgroup of \(G\), i.e.,
it is a compact Lie subgroup.
$g = \dim \mathbb{R}^n$

$K \times \mathbb{R}^n$ is simply connected

$K \times \mathbb{Z}$ is a Lie group

$G$ is a Lie group isomorphism such that $\pi$ is a Lie algebra isomorphism.

Hence, $\tilde{G} \in \tilde{K} \times \mathbb{R}^n$.

$\tilde{G}$ is not compact.

Therefore,

$\mathbb{C}$ is infinite $\iff \mathfrak{g} = \{0\}$

Hence

$\mathbb{C}$ is finite $\iff \mathfrak{g} = \{0\}$

$[\mathfrak{l}(G), \mathfrak{l}(G)] = \mathfrak{l}(G)$. 